Zajíček's theorem

Statement, proof, and some consequences of a mathematical jewel

Notes for a blackboard talk

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May 27, 2025

Atelier des doctorants LMI/LMRS

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Simplified statement to warm up

Def – c–c curve in $\mathbb{R}^2$ A set $A \subset \mathbb{R}^2$ is a c–c curve if there exists $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ two convex functions such that

$$A = \{(z, \varphi_1(z) - \varphi_2(z)) \mid z \in \mathbb{R}\}.$$

Theorem – Baby Zajíček A set $A \subset \mathbb{R}^2$ is contained in the union of countably many c–c curves if and only if there exists $\mathfrak{J} : \mathbb{R}^2 \to \mathbb{R}$ convex and not differentiable at any point of A.

If A is a c–c curve, one can take $\mathfrak{J}(x_1, x_2) = \max(\varphi_1(x_1), x_2 + \varphi_2(x_1))$, which is convex. The points where both arguments are equal are the points of A, and by formulae for the subdifferential of maxima of convex functions [Cla90, Theorem 2.8.2], $\partial\mathfrak{J}$ contains all combinations of subdifs of φ_1 (which is contained in $(\mathbb{R}, 0)$) and $\pi_2 + \varphi_2$ (contained in $(\mathbb{R}, 1)$), so not differentiable at any point of A.
The general case is left to the reader and we only care about the converse implication.

The proof of Baby Zajíček (1/3)

First observation: If a convex function $\mathfrak{J} : \mathbb{R}^2 \to \mathbb{R}$ is not differentiable at x, then either $\mathfrak{J}(x_1, \cdot)$ or $\mathfrak{J}(\cdot, x_2)$ is not differentiable from $\mathbb{R}$ to $\mathbb{R}$. (Take $v \neq w$ in the subdifferential, and assume $v_1 < w_1$. Then both $v_1, w_1 \in \partial\mathfrak{J}(\cdot, x_2)$.) Since we want a countable covering, it suffices to show that each set

$$A_1 := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \mathfrak{J}(\cdot, x_2) \text{ is not differentiable at } x_1\right\},$$
$$A_2 := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \mathfrak{J}(x_1, \cdot) \text{ is not differentiable at } x_2\right\}$$

can be covered by countably many c–c curves.
Second observation: A convex function from $\mathbb{R}$ to $\mathbb{R}$ is not differentiable iff there exists $r_1 < r_2 \in \mathbb{Q}$ in its subdifferential. So, splitting A_1 in a countable union of sets

$$A_1^{r_1, r_2} := \{(x_1, x_2) \in A_1 \mid r_i \in \partial_{x_1}\mathfrak{J}(\cdot, x_2) \text{ for } i \in \{1, 2\}\},$$

we may now reduce to the countable covering of each $A_1^{r_1, r_2}$ by c–c curves. We go on with A_1.

The proof of Baby Zajíček (2/3)

Core of the argument: Fix $r_1 < r_2 \in \mathbb{Q}$. By construction, for any $x = (x_1, x_2) \in A_1^{r_1,r_2}$, there exists $s_i^x \in \mathbb{R}$ such that $(r_i, s_i^x) \in \partial_x \mathfrak{J}$. In consequence, for all $y \in \mathbb{R}^2$,

$$\mathfrak{J}(y) \geqslant \mathfrak{J}(x) + \langle y - x, (r_i, s_i^x) \rangle = \mathfrak{J}(x) + (y_1 - x_1) r_i + (y_2 - x_2) s_i^x,$$
$$\mathfrak{J}(y) - r_i y_1 \geqslant \mathfrak{J}(x) - r_i x_1 + (y_2 - x_2) s_i^x.$$

Let $\varphi_i : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be defined by

$$\varphi_i(z) \coloneqq \sup_{(x_1,x_2) \in B_1^{r_i}} [\mathfrak{J}(x_1, x_2) - r_i x_1] + (z - x_2) s_i^x.$$

As a supremum of affine functions, φ_i is convex. We showed that $\mathfrak{J}(y_1, y_2) - r_i y_1 \geqslant \varphi_i(y_2)$ for any $y \in \mathbb{R}^2$. In the particular case where $y \in A_1^{r_1,r_2}$, there holds $\mathfrak{J}(y_1, y_2) - r_i y_1 + (y_2 - y_2) s_i^y \leqslant \varphi_i(y_2)$, so that equality holds.

The proof of Baby Zajíček (3/3)

We got to the equations $\mathfrak{J}(x) - r_1 x_1 = \varphi_1(x_2)$ and $\mathfrak{J}(x) - r_2 x_1 = \varphi_2(x_2)$ for any $(x_1, x_2) \in A_1^{r_1,r_2}$, which implies by substitution that

$$(r_2 - r_1)x_1 = \varphi_1(x_2) - \varphi_2(x_2), \qquad \text{hence} \qquad x_1 = \frac{\varphi_1(x_2) - \varphi_2(x_2)}{r_2 - r_1}.$$

Third observation: In the above, the values of $\varphi_i(x_2)$ are finite for any $(x_1, x_2) \in A_1^{r_1,r_2}$, but may be infinite elsewhere. This can be avoided by splitting $A_1^{r_1,r_2}$ according to the norm of s_i^x, and getting Lipschitz φ_i^j.

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Full statement

Def A set B is a c–c hypersurface of dimension k if, up to a permutation of coordinates,

$$B = \left\{ x = (x_1, \cdots, x_d) \in \mathbb{R}^d \;\middle|\; x_j = \varphi_1^j(x_1, \cdots, x_k) - \varphi_2^j(x_1, \cdots, x_k) \; j \in [\![k+1, d]\!] \right\}$$

for some convex functions $\varphi_1^j, \varphi_2^j : \mathbb{R}^k \to \mathbb{R}$.

Theorem – Zajíček [Zaj79] Let $\mathfrak{J} : \mathbb{R}^d \to \mathbb{R}$ be convex. For $1 \leqslant k < d$, denote $A^k := \left\{ x \in \mathbb{R}^d \mid \dim \partial_x \mathfrak{J} \geqslant d - k \right\}$. Then A^k can be covered by countably many c–c hypersurfaces of dimension k. Conversely, if $B \subset \mathbb{R}^d$ is covered by countably many c–c hypersurfaces of dim $d - k$, then $B \subset A^k[\mathfrak{J}]$ for some convex $\mathfrak{J} : \mathbb{R}^d \to \mathbb{R}$.

Proof by adaptation of Baby Zajíček's one.

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A fun consequence

Corollary – Zajíček as well Let $C \subset \mathbb{R}^d$ be a compact set. Then, the set P of points $x \in \mathbb{R}^d$ such that there exists *more than one* metric projection on B can be covered by countably many c–c hypersurfaces of dimension $n-1$.

Define

$$\mathfrak{J}(x) := |x|^2 - \min_{c \in C} |x-c|^2 = \max_{c \in C} 2\langle c, x\rangle - |c^2|.$$

Then $\mathfrak{J}$ is convex and real-valued. If x has two metric projections, then two different c realize the maximum, and one sees that $\mathfrak{J}$ is not differentiable. Hence $C \subset A^{n-1}[\mathfrak{J}]$, and the conclusion follows.

A resounding consequence

Corollary – Gigli [Gig11] The transport-regular measures in $\mathscr{P}_2(\mathbb{R}^d)$ are exactly those which give 0 mass to any c–c hypersurface of dimension $n-1$.

Path:

- A transport plan is optimal iff [marginal conditions] + [condition on the support].
- The condition on the support exactly coincides with Rockafellar's theorem describing which sets are contained in the subdifferential of a convex function.
- If [included in the subdiff of a convex function] + [gives no mass to any set of nondiff of a convex function], then must give full mass to a set where said convex function is differentiable.
- Differentiable $\implies$ gradient $\implies$ transport map!

Thank you!

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