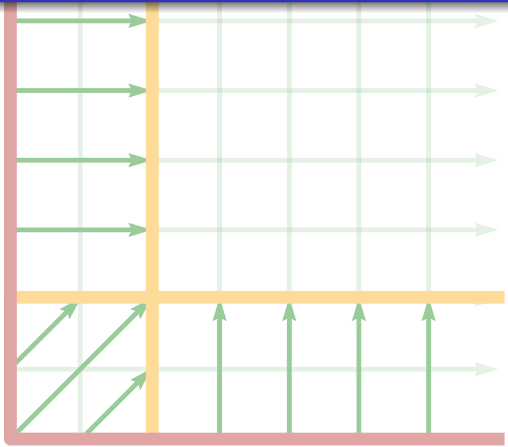
Who's who in $\mathscr{P}_2$

Towards a characterization of the geometric tangent cone to the Wasserstein space



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- For vector fields: Fix $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, and $f \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$. If $f = \nabla \varphi$ for some $\varphi \in \mathcal{C}^\infty_c$, then

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (id + hf)_\# \mu)}{h} = \|f\|_{L^2_\mu}. \tag{1}$$

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- The vector fields f for which (1) holds are "almost optimal near 0". Very close to the metric definition of the tangent cone: equivalent?

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$\mathrm{T}\,\mathbb{R}^d = \{\,(x,v) \mid x$ is a point, v is a vector $\}$. For more than one vector, $\mathrm{T}^n\,\mathbb{R}^d = \{(x, v_1, \cdots, v_n)\}$.

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In particular, $W_\mu(f_\#\mu, g_\#\mu) = \|f-g\|_{L^2_\mu}$ for any $f,g \in L^2_\mu(\mathbb{R}^d;\mathrm{T}\,\mathbb{R}^d)$.

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Fix $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. For $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$, let

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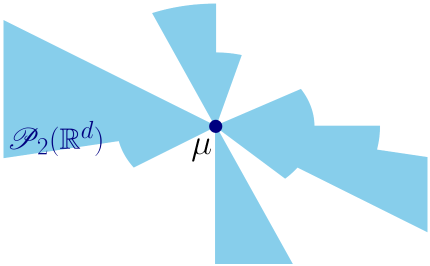

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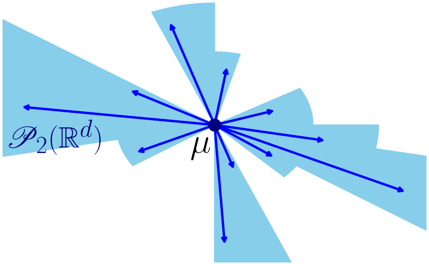

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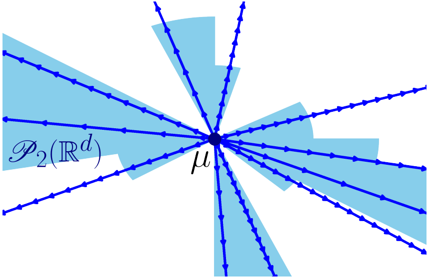

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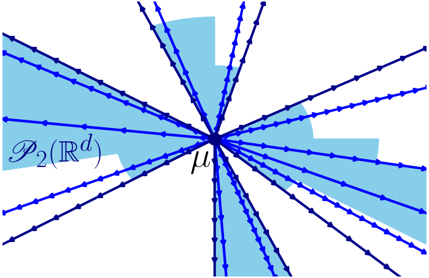

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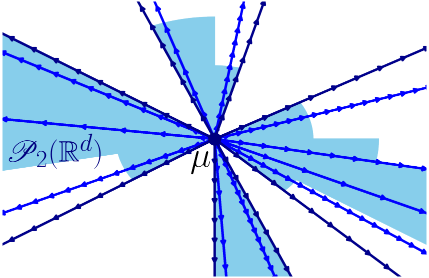

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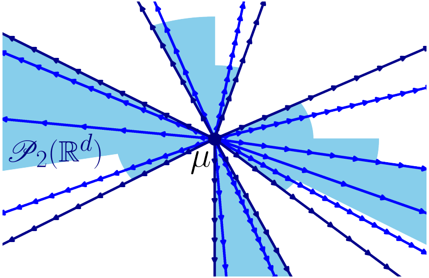

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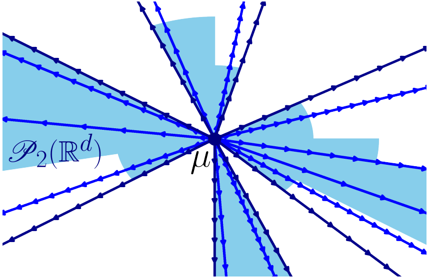

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[Gig08, Prop. 4.30] For $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$, there is a unique $\pi_T^\mu \xi$ minimizing $W_\mu(\xi, \cdot)$ over $\mathbf{Tan}_\mu$.

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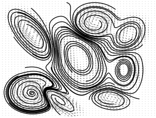

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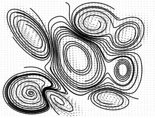

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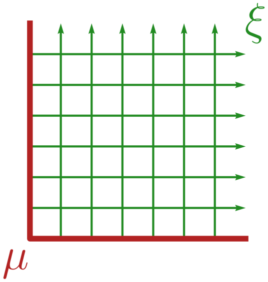
Proposition For any $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$, there is a unique $\pi^\mu_S \xi$ minimizing $W_\mu(\xi,\cdot)$ over $\mathbf{Sol}_\mu$. Moreover, $\xi = (\pi_x, \pi_v + \pi_w)_\# \alpha$ for some (optimal) $\alpha \in \Gamma_\mu(\pi^\mu_T \xi, \pi^\mu_S \xi)$. (Helmholtz-Hodge!)

Further examples

Of the tangent cone

Closing rescaled geodesics with respect to W_μ allows to "cross immediately", if on small mass.



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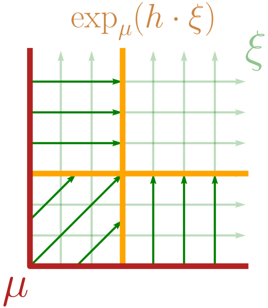

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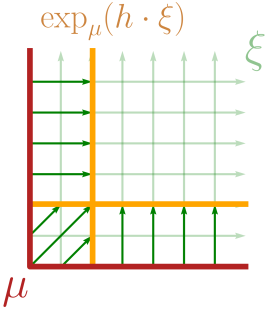

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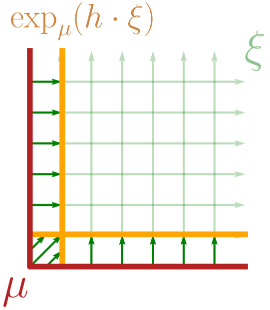

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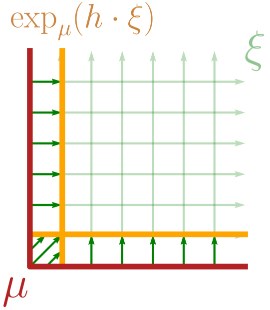

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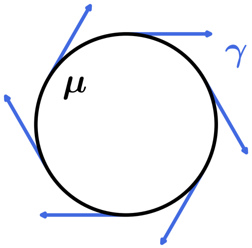

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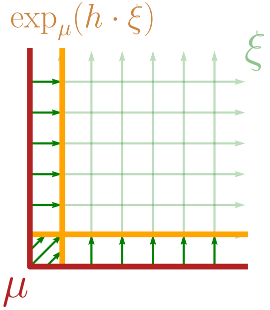

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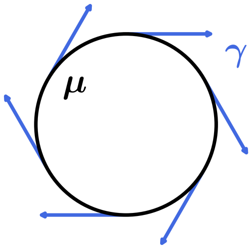

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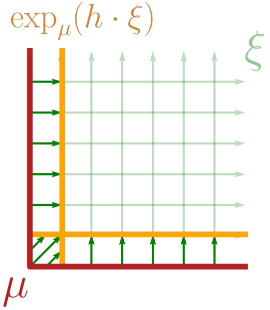

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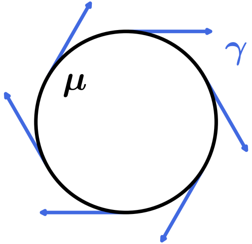

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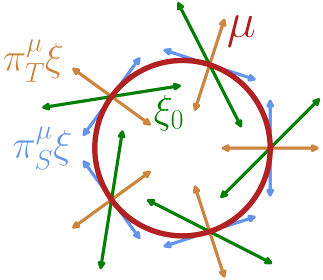

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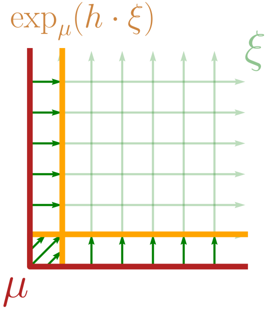

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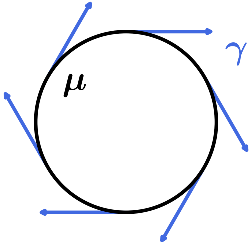

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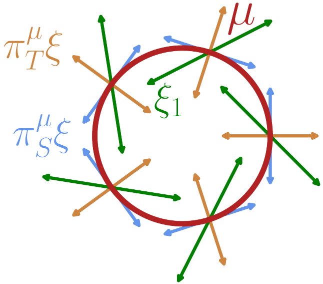

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- Directly along $h \mapsto \exp_\mu(h \cdot f_\# \mu)$: not related to the tangent cone.

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A way to win both: is it true that for all $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, and $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi_T^\mu \xi))}{h} = 0 \quad ? \qquad (\mathcal{Q})$$

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The question $(\mathcal{Q})$ is whether $\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi_T^\mu \xi))}{h} = 0$ for all $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$.

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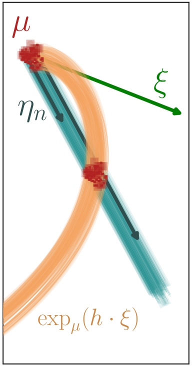
$d_{\mathcal{W},\mathrm{T}\,\mathbb{R}^d}$—topology

Theorem – Approximation in $d_{\mathcal{W},\mathrm{T}\,\mathbb{R}^d}$ Let $\xi \in \mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$. If

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h \cdot \xi))}{h} = \|\xi\|_\mu, \tag{2}$$

then there exists a sequence $(h_n)_{n\in\mathbb{N}} \searrow 0$ such that for all choices of optimal plans $\eta_n \in \Gamma_o(\mu, \exp_\mu(h_n \cdot \xi))$,

$$\lim_{n\to\infty} d_{\mathcal{W},\mathrm{T}\,\mathbb{R}^d}\left(\xi, \left(\pi_x, \frac{\pi_y - \pi_x}{h_n}\right)_{\#}\eta_n\right) = 0. \tag{3}$$



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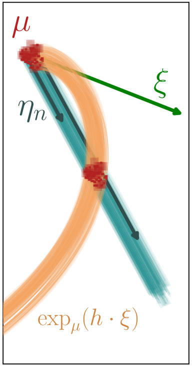

If we can replace $d_{\mathcal{W},\mathrm{T}\,\mathbb{R}^d}$ by W_μ in (3), we get that $\xi \in \mathbf{Tan}_\mu$ by definition.

Roughly, (2) forces the velocity of $h \mapsto \exp_\mu(h \cdot \xi)$ to "align" with optimal plans, enough to get (3).

Maps

Corollary – Map-induced fields Let $\gamma = f_{\#}\mu$ for some $f \in L^2_\mu(\mathbb{R}^d; \mathrm{T}\,\mathbb{R}^d)$.

If $$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h \cdot \gamma))}{h} = \|\gamma\|_\mu,$$ then $$\gamma \in \mathbf{Tan}_\mu\,.$$

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Argument: from the previous result, and the fact that the limit is induced by a map, one improves the convergence from $d_{\mathcal{W},\mathrm{T}\,\mathbb{R}^d}$ to W_μ.

General case

Proposition There exists $\mu \in \mathscr{P}_2(\mathbb{R})$ and $\zeta \in \mathbf{Sol}_\mu$ such that

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- To understand, let us detail dimension one.

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Formula for $\mathbf{Tan}_\mu$ and $\mathbf{Sol}_\mu$

In dimension one, a measure μ can be written as $\mu = m_a\mu^a + m_d\mu^d$, with $\mu^a \in \mathscr{P}_2(\mathbb{R})$ purely atomic and $\mu^d \in \mathscr{P}_2(\mathbb{R})$ diffuse (atomless).

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- Structure of solenoidal measure fields by orthogonality.

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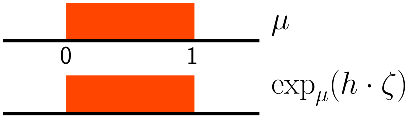

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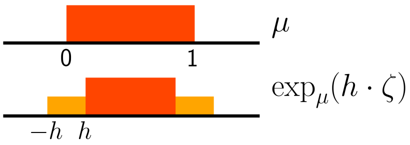

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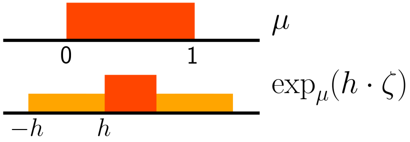

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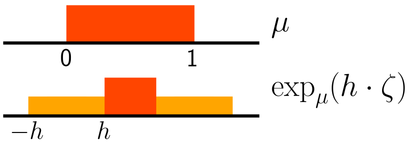

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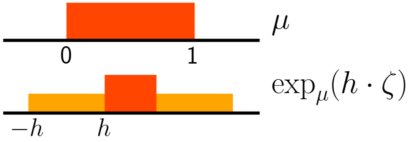

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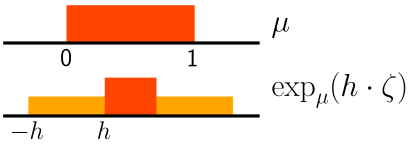

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Situation so far	maps	plans
purely atomic abs. continuous	👍	👍
Cantor part	👍	💀

A counterexample

Let $F[\mu] := \frac{1}{2}(id, -1)_{\#}\mu + \frac{1}{2}(id, 1)_{\#}\mu$. We want to construct $\mu \in \mathscr{P}_2(\mathbb{R})$ such that along $(h_n)_{n\in\mathbb{N}} \searrow 0$,

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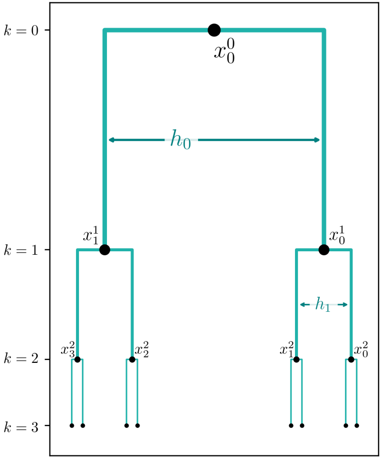
- Find μ such that $d_{\mathcal{W}}(\mu, \mu^{\text{ex}}_{h_n}) = o(h_n)$: $d_{\mathcal{W}}$−limit of a sequence of atomic measures, splitting each atom by small distances converging very fast to 0.



A counterexample

Let $F[\mu] := \frac{1}{2}(id,-1)_{\#}\mu + \frac{1}{2}(id,1)_{\#}\mu$. We want to construct $\mu \in \mathscr{P}_2(\mathbb{R})$ such that along $(h_n)_{n\in\mathbb{N}} \searrow 0$,

$$\frac{d_{\mathcal{W}}(\mu, \exp_\mu(h_n \cdot F[\mu]))}{h_n} \to_n 1 = \|F[\mu]\|_\mu.$$

For which measures does equality hold for $h_n > 0$?

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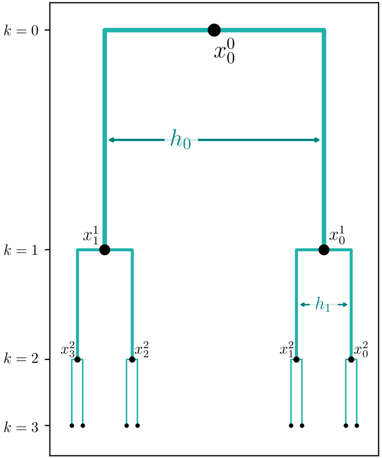
- Limit is diffuse, and $d_{\mathcal{W}}(\mu, \exp_\mu(h_n \cdot F[\mu]))/h_n \to_n 1$.



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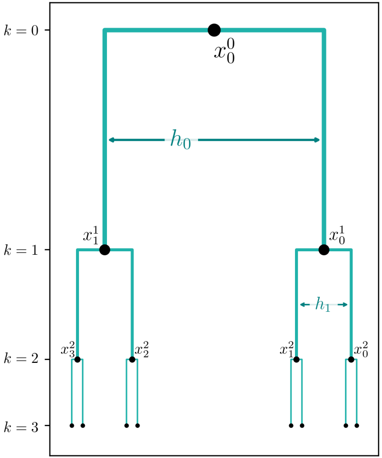
- Limit is diffuse, and $d_{\mathcal{W}}(\mu, \exp_\mu(h_n \cdot F[\mu]))/h_n \to_n 1$.
- Tweak a little the example to get the limit when $h \searrow 0$.



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- General case Get some classes for which the equivalences hold, understand what goes wrong.

Thank you!

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