

Weird measures near the edges of Otto's manifold in dimension one

Averil Aussedat
University of Pisa

March 16, 2026
OT in GMT – GMT in OT
Notes for a blackboard talk



UNIVERSITÀ DI PISA

Table of Contents

Introduction

Motivation

Old results

The problem under consideration

We care about a problem in classical OT.

Notations:

- μ a measure,
- pushforward,
- $\xi \in \mathcal{P}_2(\mathbb{T}\Omega)_\mu$ a measure field,
- W_μ distance.

Definition(s) of tangent cone

- Definition coming from manifolds: take a set of curves that lie within your manifold, for instance exponentials, and quotient them by the equivalence relation of “begin infinitesimally equivalent near 0”.
- Definition coming from Alexandrov geometry: take the set of geodesics, then the cone over it, then the “abstract” closure with respect to the distance $\limsup_{t \searrow 0} d(\gamma_t, \gamma'_t)/t$.

Basic question: is it the same in \mathcal{P}_2 ? By definition, two elements $\xi, \xi' \in \mathbf{Tan}_\mu$ must not be equivalent. So the question reformulates as follows: if $\xi \notin \mathbf{Tan}_\mu$, is it equivalent to at least one element of \mathbf{Tan}_μ ? Candidate: the projection over \mathbf{Tan}_μ with respect to W_μ (existence: Gigli 2008). Denote it $\pi^\mu \xi$.

Question:

$$\lim_{h \searrow 0} \frac{W^2(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi^\mu \xi))}{h^2} = 0 \quad ?$$

Examples of things in and out the geometric tangent cone

- If $\mu = \delta_0$, then everything.
- If $\mu = \mathcal{H}^1 \llcorner \mathcal{S}^1$, then not everything.
- If $\mu = \mathcal{H}^k \llcorner \mathcal{M}$ for a smooth surface of dimension k , then centred directions in \mathbf{Tan}_μ are going away from μ . (Lott)
- If $\mu \ll \mathcal{L}$, then only maps that are L^2 -limits of gradients (Brenier, McCann, Gigli).

Table of Contents

Introduction

Motivation

Old results

The problem under consideration

Formulation of PDEs (1/2)

Consider $-\partial_t u(t, x) + H(x, Du(t, x)) = 0$, plus terminal condition. Formulation in Alexandrov spaces with lower bound on the curvature to have plenty of semiconcave test functions, and build cylindrical functions from compositions of the squared distance. Nice theory, cool. The Wasserstein space is an Alexandrov space, can we apply it?

Problems of interest: control of population. Take trajectories solving the continuity equation:

$$\mu_t := \Phi_{t\#}\mu_0$$

for $\Phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ the flow, that satisfies $\frac{d}{dt}\Phi_t = f(\Phi_t)$; put a control over it, say Φ_t^u , and try to minimize some cost function, say $G(\mu_T^u)$. The link with the previous HJ equation is that the value function

$$V(t, \mu) := \min_u G(\mu_T^{t,u,\mu})$$

should solve the HJB equation (in the viscosity sense)

$$-\partial_t V(t, \mu) + \sup_u D_\mu V(t, \mu)(f(\mu, u)) = 0.$$

Formulation of PDEs (2/2)

Now, in the model, there is no reason that $f(\mu, u) \in \mathbf{Tan}_\mu$. And in very natural examples, it does not hold. So to apply your theory in Alexandrov spaces, you must first be able to come back to elements of the tangent cone. Since any reasonable test function will be locally Lipschitz, this would be easy if we knew that

$$\lim_{h \searrow 0} \frac{W^2(\exp_\mu(h \cdot \xi), \exp_\mu(h \cdot \pi^\mu \xi))}{h^2} = 0,$$

since then, the directional derivative w.r.t. a direction ξ coincides with that along the direction $\pi^\mu \xi \in \mathbf{Tan}_\mu$.

Problem: it is false.

Table of Contents

Introduction

Motivation

Old results

The problem under consideration

Positive results

Theorem If ξ is deterministic, i.e. of the form $(id, g)_\# \mu$ for some vector field $g \in L^2_\mu$, then the limit holds.

Saves the strategy for the PDE problem, because velocities are deterministic. In particular, in this case,

$$\xi \in \mathbf{Tan}_\mu^\perp \iff -\operatorname{div}(g\mu) = 0 \text{ distributionally} \iff \lim_{h \searrow 0} \frac{W^2(\mu, \exp_\mu(h \cdot \xi))}{h^2} = 0.$$

Not very interesting in dimension one, because the only such g is 0.

Theorem In dimension one, if μ is absolutely continuous with respect to the Lebesgue measure, or purely atomic, then the limit holds (regardless of being deterministic or not).

So regularity conditions on the measure can restore the cherished equivalence.

Very negative result

Theorem There exists μ supported on $[0, 1]$ such that for some $\xi \in \mathbf{Tan}_\mu^\perp$,

$$\limsup_{h \searrow 0} \frac{W^2(\mu, \exp_\mu(h \cdot \xi))}{h^2} = \|\xi\|_\mu^2. \quad (1)$$

This is the worst case possible!

Table of Contents

Introduction

Motivation

Old results

The problem under consideration

Statement of the problem

- First simplification: we consider the dimension one.
- Second simplification: we consider ξ as the unit symmetric measure field.
- Third simplification: we only care for the measures for which the worst case happens.

For which $\mu \in \mathcal{P}_2$ does it hold that

$$\limsup_{h \searrow 0} \frac{W^2(\mu, \mu_h)}{h^2} = 1, \quad \text{where } \mu_h := \frac{1}{2} [(id - h)_\# \mu + (id + h)_\# \mu] \quad ?$$

Some examples

- If $\mu = \delta_0$; then yes. *Drawing* In general, if μ is a finite superposition of Dirac masses, yes, and purely atomic measures also.
- If $\mu = \mathcal{L}_{[0,1]}$; then no. *Drawing* After computation, $W^2(\mu, \mu_h) \sim h^3$.
- If μ is the “uniform measure on the Cantor set”, the derivative of the Cantor staircase, then no. *Drawing* The limit does not exist, but the value oscillates somewhere between 0.5 and 0.6.

Result

Definition A measurable set $A \subset \mathbb{R}$ belongs to the class \mathcal{A} if there exist sequences $(s_n)_n \subset (0, 1)$ and $(\tau_n)_n \subset [0, 1)$ going to 0 such that for any $n \in \mathbb{N}$, and any $x \in A$,

$$\mathcal{B}(x, s_n) \cap A \subset \overline{\mathcal{B}(x, \tau_n s_n)}.$$

For instance: finite sets, some (uncountable) skinny Cantor sets (removing a proportion going to 1).

Theorem Let $\mu \in \mathcal{P}_2(\mathbb{R})$. The following assertions are equivalent:

- $\limsup_{h \searrow 0} W^2(\mu, \mu_h)/h^2 = 1$,
- for any $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $\mu(A) \geq 1 - \varepsilon$.

Idea of the proof (1/2)

- From second to first: almost explicit construction.

$$\varphi^c(u) := \inf_{v \in \mathbb{R}} \varphi(v) + |u - v|^2$$

- From first to second: dual formulation. If $\lim_{n \rightarrow \infty} W^2(\mu, \mu_{h_n})/h_n^2 = 1$ for some sequence $(h_n)_n$, then, for some Kantorovich potential φ_n ,

$$1 - O(h_n) \leq \frac{1}{h_n^2} \left(\int \varphi_n^c(y) d\mu_{h_n} - \int \varphi_n d\mu \right) = \int \underbrace{\frac{\frac{1}{2} (\varphi_n^c(x - h_n) + \varphi_n^c(x + h_n)) - \varphi_n(x)}{h_n^2}}_{\leq 1} d\mu.$$

So [the quotient](#) must be close to 1 on a set of large μ -measure.

Key argument: for any $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and $h > 0$, let

$$S := \left\{ x \in \mathbb{R} \mid \frac{\frac{1}{2} (\varphi^c(x - h) + \varphi^c(x + h)) - \varphi(x)}{h^2} \geq \gamma \right\}.$$

Idea of the proof (2/2)

Then S has gaps of size comparable h . Indeed, if $x, y \in S$ with $x \leq y$, then

$$\begin{aligned} 2\gamma &\leq \frac{\frac{1}{2}(\varphi^c(x-h) + \varphi^c(x+h)) - \varphi(x)}{h^2} + \frac{\frac{1}{2}(\varphi^c(y-h) + \varphi^c(y+h)) - \varphi(y)}{h^2} \\ &\leq \frac{\frac{1}{2}(\varphi(x) + h^2 + \varphi(y) + |x+h-y|^2) - \varphi(x)}{h^2} + \frac{\frac{1}{2}(\varphi(x) + |y-h-x|^2 + \varphi(y) + h^2) - \varphi(y)}{h^2} \\ &= 1 + \frac{|y - (x+h)|^2}{h^2}. \end{aligned}$$

$$h\sqrt{2\gamma - 1} \leq |y - (x+h)|$$

So $|y - x|$ is either smaller than ch , or larger than Ch , for c, C depending only on γ .

Now classical trick of taking intersections of sets of mass larger than $1 - \varepsilon 2^{-(k+1)}$ and recover a set of mass larger than $1 - \varepsilon$. □

Corollary on the classification problem

Corollary Let $\mu \in \mathcal{P}_2(\mathbb{R})$. The following assertions are equivalent:

- $\xi \in \mathbf{Tan}_\mu$ if and only if $\limsup_{h \searrow 0} W^2(\mu, \exp_\mu(h \cdot \xi))/h^2 = \|\xi\|_\mu^2$,
- $\mu^d(A) = 0$ for any $A \in \mathcal{A}$, where μ^d is the diffuse part of μ (removing atoms).

!! Does not give answers for the general problem.