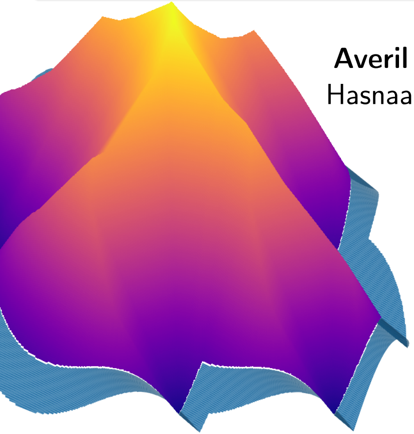
Viscosity solutions in the Wasserstein space

Link between test functions and semidifferentials



Averil Prost (LMI, INSA Rouen Normandie)
Hasnaa Zidani (LMI, INSA Rouen Normandie)

March 27, 2024
SMAI MODE Days

Aim of the talk

Consider a Hamilton-Jacobi equation

$$H\left(\mu, D_\mu V(\mu)\right) = 0 \quad \mu \in \Omega, \qquad V(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{1}$$

Aim of the talk

Consider a Hamilton-Jacobi equation

$$H(\mu, D_\mu V(\mu)) = 0 \quad \mu \in \Omega, \qquad V(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{1}$$

Here

- μ is a measure, Ω an open set of the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$.

Aim of the talk

Consider a Hamilton-Jacobi equation

$$H\left(\mu, D_\mu V(\mu)\right) = 0 \quad \mu \in \Omega, \qquad V(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{1}$$

Here

- μ is a measure, Ω an open set of the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$.
- $D_\mu V(\mu)$ is the application of the directional derivatives.

Aim of the talk

Consider a Hamilton-Jacobi equation

$$H(\mu, D_\mu V(\mu)) = 0 \quad \mu \in \Omega, \qquad V(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{1}$$

Here

- μ is a measure, Ω an open set of the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$.
- $D_\mu V(\mu)$ is the application of the directional derivatives.

Our aim Compare two notions of viscosity solutions for (1).

Table of Contents

First-order Hamilton-Jacobi equations

Nonlinear first-order equations in an open set $\Omega \subset \mathbb{R}^d$:

$$H\left(x, \nabla u(x)\right) = 0 \quad x \in \Omega, \qquad u(x) = \mathfrak{J}(x) \quad x \in \partial\Omega.$$

First-order Hamilton-Jacobi equations

Nonlinear first-order equations in an open set $\Omega \subset \mathbb{R}^d$:

$$H\left(x, \nabla u(x)\right) = 0 \quad x \in \Omega, \qquad u(x) = \mathfrak{J}(x) \quad x \in \partial\Omega.$$

Classical examples include

- the Eikonal equation $H\left(x, p\right) = \left|p\right|$,

First-order Hamilton-Jacobi equations

Nonlinear first-order equations in an open set $\Omega \subset \mathbb{R}^d$:

$$H\left(x, \nabla u(x)\right) = 0 \quad x \in \Omega, \qquad u(x) = \mathfrak{J}(x) \quad x \in \partial\Omega.$$

Classical examples include

- the Eikonal equation $H\left(x, p\right) = \left|p\right|$,
- HJB equations $H\left(x, (p_t, p_x)\right) = -p_t + \sup_{b \in F[x]} - \langle p_x, b \rangle$,

First-order Hamilton-Jacobi equations

Nonlinear first-order equations in an open set $\Omega \subset \mathbb{R}^d$:

$$H\left(x, \nabla u(x)\right) = 0 \quad x \in \Omega, \qquad u(x) = \mathfrak{J}(x) \quad x \in \partial\Omega.$$

Classical examples include

- the Eikonal equation $H\left(x, p\right) = \left|p\right|$,
- HJB equations $H\left(x, (p_t, p_x)\right) = -p_t + \sup_{b \in F[x]} - \langle p_x, b \rangle$,
- Ishii equations $H\left(x, p\right) = \sup_{a \in A} \inf_{b \in B} - \langle p, f(x, a, b) \rangle$...

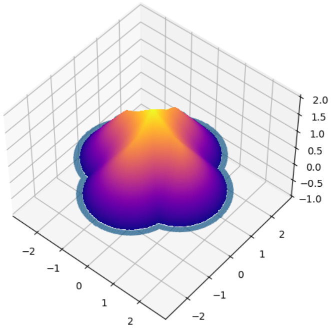
First-order Hamilton-Jacobi equations

Nonlinear first-order equations in an open set $\Omega \subset \mathbb{R}^d$:

$$H(x, \nabla u(x)) = 0 \quad x \in \Omega, \qquad u(x) = \mathfrak{J}(x) \quad x \in \partial\Omega.$$

Classical examples include

- the Eikonal equation $H(x, p) = |p|$,
- HJB equations $H(x, (p_t, p_x)) = -p_t + \sup_{b \in F[x]} - \langle p_x, b \rangle$,
- Ishii equations $H(x, p) = \sup_{a \in A} \inf_{b \in B} - \langle p, f(x, a, b) \rangle$...



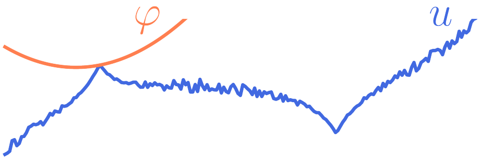
Usually nonsmooth solutions, as the distance to the boundary.
Need for an adapted notion of weak solutions.

Viscosity solutions

In $\mathbb{R}^d$, viscosity solutions are equivalently defined using

- smooth test functions:

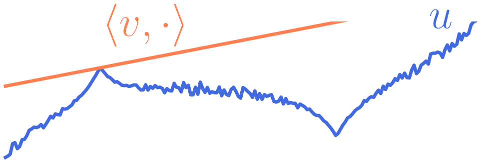
u is a **subsolution** if it is **u.s.c**, satisfies $u \leqslant \mathfrak{J}$, and if whenever $\varphi \in \mathcal{C}^1$ is such that $u - \varphi$ reaches a **maximum** at x,



there holds $H(x, \nabla\varphi(x)) \leqslant 0$.

- sub and superdifferentials:

u is a **subsolution** if it is **u.s.c**, satisfies $u \leqslant \mathfrak{J}$, and if whenever a vector v belongs to the **superdifferential** of u at x,



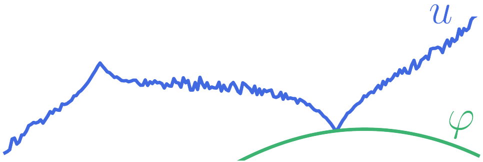
there holds $H(x, v) \leqslant 0$.

Viscosity solutions

In $\mathbb{R}^d$, viscosity solutions are equivalently defined using

- smooth test functions:

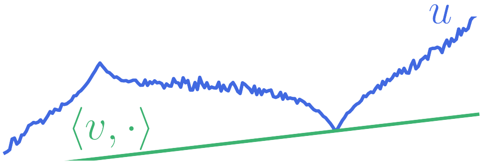
u is a **supersolution** if it is **l.s.c**, satisfies $u \geqslant \mathfrak{J}$, and if whenever $\varphi \in \mathcal{C}^1$ is such that $u - \varphi$ reaches a **minimum** at x,



there holds $H(x, \nabla\varphi(x)) \geqslant 0$.

- sub and superdifferentials:

u is a **supersolution** if it is **l.s.c**, satisfies $u \geqslant \mathfrak{J}$, and if whenever a vector v belongs to the **subdifferential** of u at x,

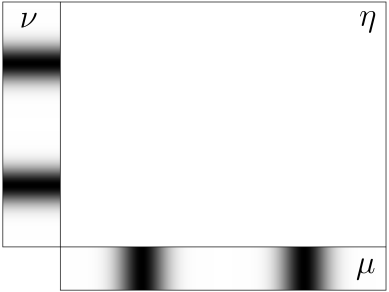


there holds $H(x, v) \geqslant 0$.

Table of Contents

The mathematical definition

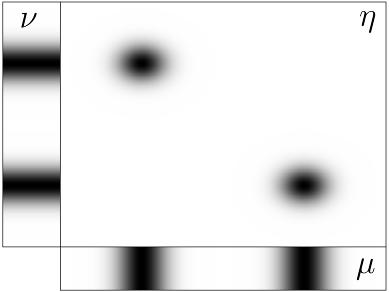
Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures,



The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

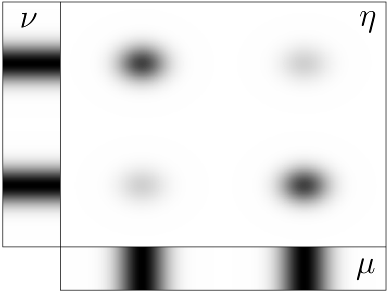
$$\Gamma(\mu,\nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$



The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

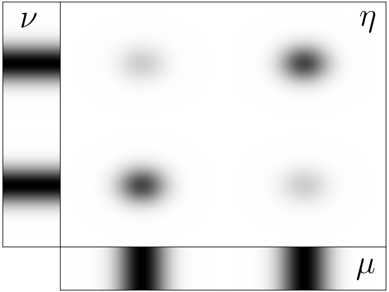
$$\Gamma(\mu,\nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$



The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

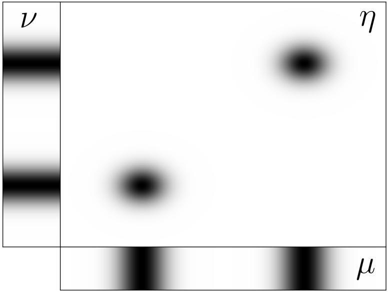
$$\Gamma(\mu, \nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$



The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

$$\Gamma(\mu,\nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$



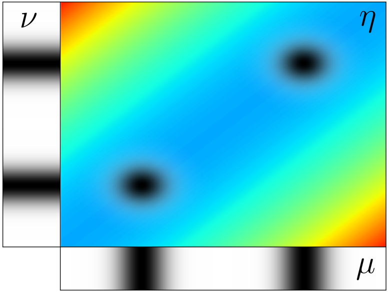
The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

$$\Gamma(\mu,\nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$

the squared Wasserstein distance by

$$d_{\mathcal{W}}^2(\mu,\nu) := \inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^2 \, d\eta(x,y).$$



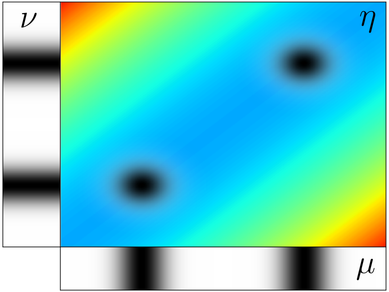
The mathematical definition

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures, and denote the set of transport plans by

$$\Gamma(\mu,\nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\},$$

the squared Wasserstein distance by

$$d_{\mathcal{W}}^2(\mu,\nu) := \inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^2 \, d\eta(x,y).$$



Def We call **Wasserstein space** the set $\mathscr{P}_2(\mathbb{R}^d) := \left\{ \mu \in \mathscr{P}(\mathbb{R}^d) \;\middle|\; d_{\mathcal{W}}(\mu,\delta_0) < \infty \right\}$, endowed with the distance $d_{\mathcal{W}}$.

The artistic definition

Jean-Olivier Héron, 1997

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$.

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$. Then lift $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $U : L^2_{\mathbb{P}}(E; \mathbb{R}^d) \to \mathbb{R}$ by

$$U(X) := u(X \# \mathbb{P})$$

to use the differentiability in the Hilbert $L^2_{\mathbb{P}}(E; \mathbb{R}^d)$ [Lio07, CD18].

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$. Then lift $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $U : L^2_{\mathbb{P}}(E; \mathbb{R}^d) \to \mathbb{R}$ by

$$U(X) := u(X \# \mathbb{P})$$

to use the differentiability in the Hilbert $L^2_{\mathbb{P}}(E; \mathbb{R}^d)$ [Lio07, CD18]. Allows to define $\mathcal{C}^1$ maps. **Problem: find them!** Mollification procedures [CM23] or penalization [DJS23].

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$. Then lift $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $U : L^2_{\mathbb{P}}(E; \mathbb{R}^d) \to \mathbb{R}$ by

$$U(X) := u(X\#\mathbb{P})$$

  to use the differentiability in the Hilbert $L^2_{\mathbb{P}}(E; \mathbb{R}^d)$ [Lio07, CD18]. Allows to define $\mathcal{C}^1$ maps. **Problem: find them!** Mollification procedures [CM23] or penalization [DJS23].
- **Semidifferentials**: defined as elements of $L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, used in [GNT08], in the series [MQ18, JMQ20, JMQ22], extension to $\mathscr{P}_2$(graphs) in [GMŚ23].

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$. Then lift $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $U : L^2_{\mathbb{P}}(E; \mathbb{R}^d) \to \mathbb{R}$ by

$$U(X) := u(X \# \mathbb{P})$$

  to use the differentiability in the Hilbert $L^2_{\mathbb{P}}(E; \mathbb{R}^d)$ [Lio07, CD18]. Allows to define $\mathcal{C}^1$ maps. **Problem: find them!** Mollification procedures [CM23] or penalization [DJS23].
- **Semidifferentials**: defined as elements of $L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, used in [GNT08], in the series [MQ18, JMQ20, JMQ22], extension to $\mathscr{P}_2$(graphs) in [GMŚ23]. Proved to be equivalent to Lions differentiability in [GT19].

Differentiability in the Wasserstein space

- **Lions differentiability**: recast μ as the law of some random variable $X : (E, \mathcal{E}, \mathbb{P}) \to \mathbb{R}^d$. Then lift $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ in $U : L^2_{\mathbb{P}}(E; \mathbb{R}^d) \to \mathbb{R}$ by

$$U(X) := u(X\#\mathbb{P})$$

  to use the differentiability in the Hilbert $L^2_{\mathbb{P}}(E; \mathbb{R}^d)$ [Lio07, CD18]. Allows to define $\mathcal{C}^1$ maps. **Problem: find them!** Mollification procedures [CM23] or penalization [DJS23].
- **Semidifferentials**: defined as elements of $L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, used in [GNT08], in the series [MQ18, JMQ20, JMQ22], extension to $\mathscr{P}_2$(graphs) in [GMŚ23]. Proved to be equivalent to Lions differentiability in [GT19].
- Insights from **viscosity in metric spaces** for Eikonal-type equations: [AF14] defines generalized semidifferentials, to obtain consistency with their notion based on metric slope.

Directional derivatives ([JJZ, Jer22, JPZ23])

Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\ p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

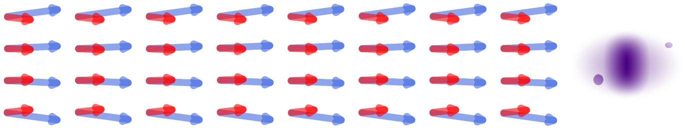
Directional derivatives ([JJZ, Jer22, JPZ23])

Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\ p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu\varphi : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu\varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$



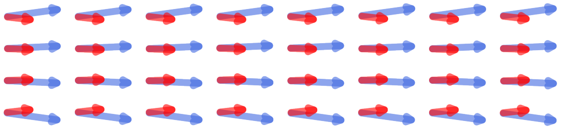
Directional derivatives ([JJZ, Jer22, JPZ23])

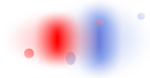
Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\; p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu\varphi : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu\varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$





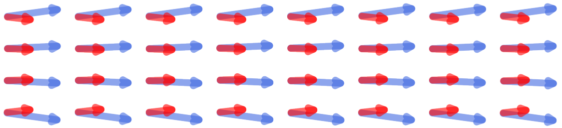
Directional derivatives ([JJZ, Jer22, JPZ23])

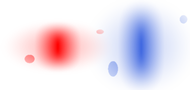
Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\ p : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu\varphi : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu\varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$





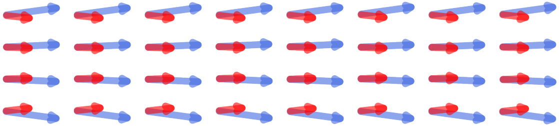
Directional derivatives ([JJZ, Jer22, JPZ23])

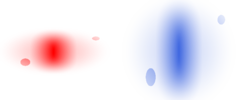
Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\; p : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu\varphi : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu\varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$





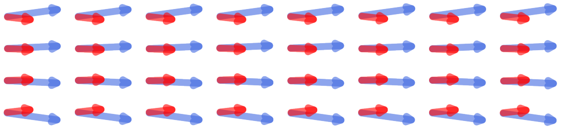
Directional derivatives ([JJZ, Jer22, JPZ23])

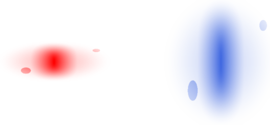
Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\; p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu \varphi : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu\varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$





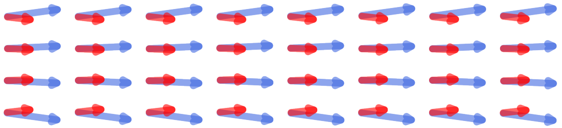
Directional derivatives ([JJZ, Jer22, JPZ23])

Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\; p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu \varphi : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu \varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$



Directional derivatives ([JJZ, Jer22, JPZ23])

Define the Hamiltonian H as $H : D(H) \subset \mathbb{T} \to \mathbb{R}$, where

$$\mathbb{T} := \left\{ (\mu, p) \;\middle|\; \mu \in \mathscr{P}_2(\mathbb{R}^d),\ p : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R} \text{ sufficiently regular} \right\}.$$

Typically, p is the application of directional derivatives of a function $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, given as

$$D_\mu \varphi : \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu \to \mathbb{R}, \qquad D_\mu \varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}.$$

For instance, a control Hamiltonian writes

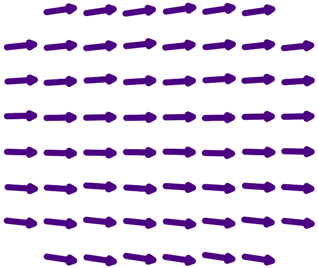
$$H(\mu, p) := \sup_{\xi \in F[\mu]} -p(\xi).$$

Table of Contents

Tangent cones

Classical "regular" tangent cone:

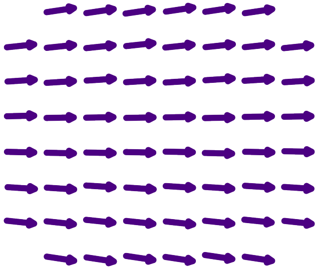
$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu}.$$



Tangent cones

Classical "regular" tangent cone:

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu}.$$



Let $\overrightarrow{\mu\nu} \subset \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ be the **set** of initial velocities of geodesics from μ to ν. Denote

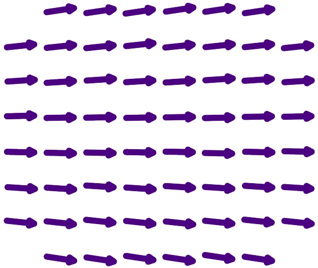
$$W^2_\mu(\xi,\zeta) := \int_{x\in\mathbb{R}^d} d^2_{\mathcal{W},\mathrm{T}_x\mathbb{R}^d}(\xi_x,\zeta_x)\, d\mu(x)$$

a generalization of the L^2_μ distance.

Tangent cones

Classical "regular" tangent cone:

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu}.$$



Let $\overrightarrow{\mu\nu} \subset \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$ be the **set** of initial velocities of geodesics from μ to ν. Denote

$$W_\mu^2(\xi,\zeta) := \int_{x\in\mathbb{R}^d} d^2_{\mathcal{W},\mathsf{T}_x\mathbb{R}^d}(\xi_x,\zeta_x)\, d\mu(x)$$

a generalization of the L^2_μ distance.

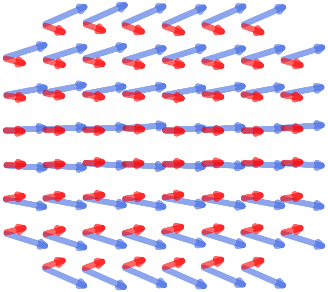
Def The generalized tangent cone is

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\mathbb{R}^+ \cdot \left\{\overrightarrow{\mu\nu} \;\middle|\; \nu \in \mathscr{P}_2(\mathbb{R}^d)\right\}}^{W_\mu}$$

Tangent cones

Classical "regular" tangent cone:

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu}.$$



Let $\overrightarrow{\mu\nu} \subset \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$ be the **set** of initial velocities of geodesics from μ to ν. Denote

$$W_\mu^2(\xi,\zeta) := \int_{x\in\mathbb{R}^d} d^2_{\mathcal{W},\mathsf{T}_x\mathbb{R}^d}(\xi_x,\zeta_x)\,d\mu(x)$$

a generalization of the L^2_μ distance.

Def The generalized tangent cone is

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\mathbb{R}^+ \cdot \left\{\overrightarrow{\mu\nu} \;\middle|\; \nu \in \mathscr{P}_2(\mathbb{R}^d)\right\}}^{W_\mu}$$

Pseudo scalar products

Def – [Gig08] Denote $\|\xi\|_\mu = W_\mu(\xi, 0_\mu)$. For any $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, define

$$\langle \xi, \zeta \rangle_\mu^+ := \frac{1}{2}\left[\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi, \zeta)\right], \qquad \langle \xi, \zeta \rangle_\mu^- := -\langle \xi, -\zeta \rangle_\mu^+ .$$

Pseudo scalar products

Def – [Gig08] Denote $\|\xi\|_\mu = W_\mu(\xi, 0_\mu)$. For any $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, define

$$\langle \xi, \zeta \rangle_\mu^+ := \frac{1}{2}\left[\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi,\zeta)\right], \qquad \langle \xi, \zeta \rangle_\mu^- := -\langle \xi, -\zeta \rangle_\mu^+ .$$

If $\xi = f\#\mu$ and $\zeta = g\#\mu$ for $f, g \in L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, then

$$\langle \xi, \zeta \rangle_\mu^+ = \langle \xi, \zeta \rangle_\mu^- = \langle f, g \rangle_{L^2_\mu} .$$

In general, $\langle \xi, \zeta \rangle_\mu^- \leqslant \langle \xi, \zeta \rangle_\mu^+$.

Pseudo scalar products

Def – [Gig08] Denote $\|\xi\|_\mu = W_\mu(\xi, 0_\mu)$. For any $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, define

$$\langle \xi, \zeta \rangle_\mu^+ := \frac{1}{2}\left[\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi,\zeta)\right], \qquad \langle \xi, \zeta \rangle_\mu^- := -\langle \xi, -\zeta \rangle_\mu^+ .$$

If $\xi = f\#\mu$ and $\zeta = g\#\mu$ for $f, g \in L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, then

$$\langle \xi, \zeta \rangle_\mu^+ = \langle \xi, \zeta \rangle_\mu^- = \langle f, g \rangle_{L^2_\mu} .$$

In general, $\langle \xi, \zeta \rangle_\mu^- \leqslant \langle \xi, \zeta \rangle_\mu^+$. For instance, if $\xi = \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)}$ in dimension 1, then

$$\langle \xi, \xi \rangle_\mu^+ = \|\xi\|_\mu^2 = 1 \qquad \text{but} \qquad \langle \xi, \xi \rangle_\mu^- = -1.$$

Semidifferentials

Def – Superdifferential Let $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. An element $\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ belongs to the superdifferential of φ at μ, denoted $\partial^+_\mu \varphi$, if for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\varphi(\nu) - \varphi(\mu) \leqslant \inf_{\eta \in \overrightarrow{\mu\nu}} \langle \xi, \eta \rangle^-_\mu + o\left(d_{\mathcal{W}}(\mu, \nu)\right).$$

Semidifferentials

Def – Superdifferential Let $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. An element $\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ belongs to the superdifferential of φ at μ, denoted $\partial_\mu^+ \varphi$, if for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\varphi(\nu) - \varphi(\mu) \leqslant \inf_{\eta \in \overrightarrow{\mu\nu}} \langle \xi, \eta \rangle_\mu^- + o\left(d_{\mathcal{W}}(\mu,\nu)\right).$$

Def – Subdifferential Let $\varphi : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$. An element $\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ belongs to the subdifferential of φ at μ, denoted $\partial_\mu^- \varphi$, if for all $\nu \in \mathscr{P}_2(\mathbb{R}^d)$,

$$\varphi(\nu) - \varphi(\mu) \geqslant \sup_{\eta \in \overrightarrow{\mu\nu}} \langle \xi, \eta \rangle_\mu^+ + o\left(d_{\mathcal{W}}(\mu,\nu)\right).$$

A good set of test functions

Def – Test functions For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, define

$$\mathscr{T}_{+,\mu} := \left\{ \varphi : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} \varphi \text{ is lower semicontinuous, directionally differentiable at } \mu, \\ \partial_\mu^+ \varphi \text{ is nonempty, bounded and } D_\mu \varphi(\mu)(\cdot) = \inf_{\zeta \in \partial_\mu^+ \varphi} \langle \cdot, \zeta \rangle_\mu^- \,. \end{array} \right\}.$$

A good set of test functions

Def – Test functions For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, define

$$\mathscr{T}_{+,\mu} := \left\{ \varphi : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} \varphi \text{ is lower semicontinuous, directionally differentiable at } \mu, \\ \partial_\mu^+ \varphi \text{ is nonempty, bounded and } D_\mu\varphi(\mu)(\cdot) = \inf_{\zeta \in \partial_\mu^+ \varphi} \langle \cdot, \zeta \rangle_\mu^- . \end{array} \right\}.$$

Similarly, $\mathscr{T}_{-,\mu} := -\mathscr{T}_{+,\mu}$.

A good set of test functions

Def – Test functions For any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, define

$$\mathscr{T}_{+,\mu} := \left\{ \varphi : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{c} \varphi \text{ is lower semicontinuous, directionally differentiable at } \mu, \\ \partial_\mu^+ \varphi \text{ is nonempty, bounded and } D_\mu\varphi(\mu)(\cdot) = \inf_{\zeta \in \partial_\mu^+\varphi} \langle \cdot, \zeta \rangle_\mu^- . \end{array} \right\}.$$

Similarly, $\mathscr{T}_{-,\mu} := -\mathscr{T}_{+,\mu}$.

For instance, if $\mu, \sigma \in \mathscr{P}_2(\mathbb{R}^d)$ and $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ are fixed,

$$\nu \mapsto d_{\mathcal{W}}^2(\nu, \sigma) \qquad \text{and} \qquad \nu \mapsto \inf_{\eta \in \overrightarrow{\mu\nu}} \langle \xi, \eta \rangle_\mu^-$$

belong to $\mathscr{T}_{+,\mu}$.

Table of Contents

Precise definitions

Consider the HJB equation

$$H(\mu, D_\mu u(\mu)) = 0 \quad \mu \in \Omega, \qquad u(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{2}$$

Def – Using test functions

A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a subsolution of (2) if it is **u.s.c**, if $u \leqslant \mathfrak{J}$ over $\partial\Omega$, and if for any μ and $\varphi \in \mathscr{T}_{+,\mu}$ such that $u - \varphi$ reaches a **maximum** at μ,

$$H(\mu, D_\mu \varphi) \leqslant 0.$$

Def – Using semidifferentials

A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a subsolution of (2) if it is **u.s.c**, if $u \leqslant \mathfrak{J}$ over $\partial\Omega$, and if for any element $\xi \in \partial_\mu^+ u$,

$$H(\mu, \langle \xi, \cdot \rangle_\mu^-) \leqslant 0.$$

A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a viscosity solution of (2) if it is both a sub and a supersolution.

Precise definitions

Consider the HJB equation

$$H(\mu, D_\mu u(\mu)) = 0 \quad \mu \in \Omega, \qquad u(\mu) = \mathfrak{J}(\mu) \quad \mu \in \partial\Omega. \tag{2}$$

Def – Using test functions
A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a subsolution of (2) if it is **l.s.c**, if $u \geqslant \mathfrak{J}$ over $\partial\Omega$, and if for any μ and $\varphi \in \mathscr{T}_{-,\mu}$ such that $u - \varphi$ reaches a **minimum** at μ,

$$H(\mu, D_\mu \varphi) \geqslant 0.$$

Def – Using semidifferentials
A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a subsolution of (2) if it is **l.s.c**, if $u \geqslant \mathfrak{J}$ over $\partial\Omega$, and if for any element $\xi \in \partial_\mu^- u$,

$$H(\mu, \langle \xi, \cdot \rangle_\mu^+) \geqslant 0.$$

A map $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a viscosity solution of (2) if it is both a sub and a supersolution.

Equivalence result

Theorem 1 Assume that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, any $\varphi \in \mathscr{T}_{+,\mu}$ and $\psi \in \mathscr{T}_{-,\mu}$,

$$H(\mu, D_\mu\varphi) \leqslant \sup_{\xi\in\partial^+_\mu\varphi} H(\mu, \langle\xi,\cdot\rangle^-_\mu), \quad \text{and} \quad H(\mu, D_\mu\psi) \geqslant \inf_{\xi\in\partial^-_\mu\psi} H(\mu, \langle\xi,\cdot\rangle^+_\mu). \quad (3)$$

Then both definitions are equivalent, in the sense that they share the same semisolutions.

Equivalence result

Theorem 1 Assume that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, any $\varphi \in \mathscr{T}_{+,\mu}$ and $\psi \in \mathscr{T}_{-,\mu}$,

$$H(\mu, D_\mu \varphi) \leqslant \sup_{\xi \in \partial_\mu^+ \varphi} H(\mu, \langle \xi, \cdot \rangle_\mu^-), \quad \text{and} \quad H(\mu, D_\mu \psi) \geqslant \inf_{\xi \in \partial_\mu^- \psi} H(\mu, \langle \xi, \cdot \rangle_\mu^+). \quad (3)$$

Then both definitions are equivalent, in the sense that they share the same semisolutions.

- Condition (3) is trivial if φ, ψ are $\mathcal{C}^1$ in the sense of Lions differentiability.

Equivalence result

Theorem 1 Assume that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, any $\varphi \in \mathscr{T}_{+,\mu}$ and $\psi \in \mathscr{T}_{-,\mu}$,

$$H(\mu, D_\mu\varphi) \leqslant \sup_{\xi \in \partial_\mu^+ \varphi} H(\mu, \langle \xi, \cdot \rangle_\mu^-), \quad \text{and} \quad H(\mu, D_\mu\psi) \geqslant \inf_{\xi \in \partial_\mu^- \psi} H(\mu, \langle \xi, \cdot \rangle_\mu^+). \quad (3)$$

Then both definitions are equivalent, in the sense that they share the same semisolutions.

- Condition (3) is trivial if φ, ψ are $\mathcal{C}^1$ in the sense of Lions differentiability.
- In the case of control problems with Lip. dynamic, a strong* comparison principle brings existence and uniqueness of the solution [JPZ23].

*with an adapted notion of semicontinuity

Equivalence result

Theorem 1 Assume that for any $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, any $\varphi \in \mathscr{T}_{+,\mu}$ and $\psi \in \mathscr{T}_{-,\mu}$,

$$H(\mu, D_\mu\varphi) \leqslant \sup_{\xi \in \partial_\mu^+ \varphi} H(\mu, \langle \xi, \cdot \rangle_\mu^-), \quad \text{and} \quad H(\mu, D_\mu\psi) \geqslant \inf_{\xi \in \partial_\mu^- \psi} H(\mu, \langle \xi, \cdot \rangle_\mu^+). \tag{3}$$

Then both definitions are equivalent, in the sense that they share the same semisolutions.

- Condition (3) is trivial if φ, ψ are $\mathcal{C}^1$ in the sense of Lions differentiability.
- In the case of control problems with Lip. dynamic, a strong* comparison principle brings existence and uniqueness of the solution [JPZ23].
- Proof by construction of a test function on one side, and using (3) on the other side.

*with an adapted notion of semicontinuity

Examples of application

- **Eikonal-type equations** Let $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing.

$$H(\mu, p) := \kappa(|p|_\mu), \qquad \text{where} \qquad |p|_\mu = \sup_{\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d), \|\xi\|_\mu = 1} |p(\xi)| \,.$$

- **"Concave-convex Hamiltonians"** Let F_1, $F_2 : \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ be set-valued maps with nonempty, horizontally convex and compact images in $(\mathbf{Tan}_\mu, W_\mu)$.

$$H(\mu, p) := \sup_{\xi_1 \in F_1[\mu]} -p(\xi_1) + \inf_{\xi_2 \in F_2[\mu]} -p(\xi_2),$$

Conclusion

Conclusion

- Possibility to use explicit test functions built from the squared Wasserstein distance or pseudo scalar products.

Conclusion

Conclusion

- Possibility to use explicit test functions built from the squared Wasserstein distance or pseudo scalar products.
- Equivalence between two notions of viscosity solutions under an explicit condition over the Hamiltonian.

Conclusion

Conclusion

- Possibility to use explicit test functions built from the squared Wasserstein distance or pseudo scalar products.
- Equivalence between two notions of viscosity solutions under an explicit condition over the Hamiltonian.

Perspectives

- Extension over $\mathscr{P}_2(\mathcal{N})$, where $\mathcal{N}$ is not Hilbertian (network structure).

Conclusion

Conclusion

- Possibility to use explicit test functions built from the squared Wasserstein distance or pseudo scalar products.
- Equivalence between two notions of viscosity solutions under an explicit condition over the Hamiltonian.

Perspectives

- Extension over $\mathscr{P}_2(\mathcal{N})$, where $\mathcal{N}$ is not Hilbertian (network structure).
- Link with Lions differentiability?

Thank you!

[AF14] Luigi Ambrosio and Jin Feng.
On a class of first order Hamilton–Jacobi equations in metric spaces.
Journal of Differential Equations, 256(7):2194–2245, April 2014.

[CD18] René Carmona and François Delarue.
Probabilistic Theory of Mean Field Games with Applications I, volume 83 of *Probability Theory and Stochastic Modelling*.
Springer International Publishing, 2018.

[CM23] Andrea Cosso and Mattia Martini.
On smooth approximations in the Wasserstein space.
Electronic Communications in Probability, 28(none):1–11, January 2023.

[DJS23] Samuel Daudin, Joe Jackson, and Benjamin Seeger.
Well-posedness of Hamilton-Jacobi equations in the Wasserstein space: Non-convex Hamiltonians and common noise, December 2023.
Preprint (arXiv:2312.02324).

[Gig08] Nicola Gigli.
On the Geometry of the Space of Probability Measures Endowed with the Quadratic Optimal Transport Distance.
PhD thesis, Scuola Normale Superiore di Pisa, Pisa, 2008.

[GMŚ23] Wilfrid Gangbo, Chenchen Mou, and Andrzej Święch.
Well-posedness for Hamilton-Jacobi equations on the Wasserstein space on graphs.
2023.

[GNT08] Wilfrid Gangbo, Truyen Nguyen, and Adrian Tudorascu.
Hamilton-Jacobi Equations in the Wasserstein Space.
Methods and Applications of Analysis, 15(2):155–184, 2008.

[GT19] Wilfrid Gangbo and Adrian Tudorascu.
On differentiability in the Wasserstein space and well-posedness for Hamilton–Jacobi equations.
Journal de Mathématiques Pures et Appliquées, 125:119–174, May 2019.

[Jer22] Othmane Jerhaoui.
Viscosity Theory of First Order Hamilton Jacobi Equations in Some Metric Spaces.
PhD thesis, Institut Polytechnique de Paris, Paris, 2022.

[JJZ] Frédéric Jean, Othmane Jerhaoui, and Hasnaa Zidani.
Deterministic optimal control on Riemannian manifolds under probability knowledge of the initial condition.
Preprint, available at https://ensta-paris.hal.science/hal-03564787.

[JMQ20] Chloé Jimenez, Antonio Marigonda, and Marc Quincampoix.
Optimal control of multiagent systems in the Wasserstein space.
Calculus of Variations and Partial Differential Equations, 59, March 2020.

[JMQ22] Chloé Jimenez, Antonio Marigonda, and Marc Quincampoix.
Dynamical systems and Hamilton-Jacobi-Bellman equations on the Wasserstein space and their L^2 representations.
Journal of Mathematical Analysis (SIMA), 2022.
(In press).

[JPZ23] Othmane Jerhaoui, Averil Prost, and Hasnaa Zidani.
Viscosity solutions of centralized control problems in measure spaces, 2023.
Preprint, available at https://hal.science/hal-04335852.

[Lio07] Pierre-Louis Lions.
Jeux à champ moyen, 2006/2007.
Conférences au Collège de France.

[MQ18] Antonio Marigonda and Marc Quincampoix.
Mayer control problem with probabilistic uncertainty on initial positions.
Journal of Differential Equations, 264(5):3212–3252, March 2018.