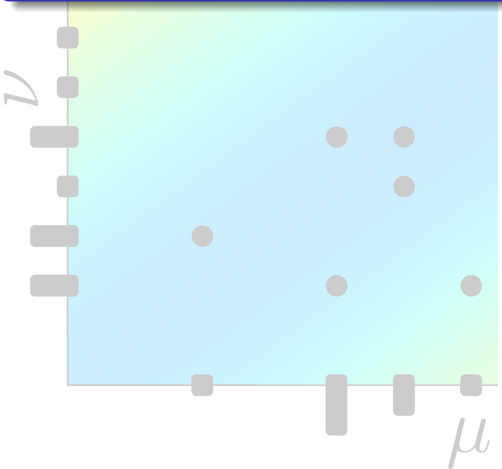
Thinking horizontally

Control problems with possibly infinite cost in the Wasserstein space



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Applied math seminar, LMJL Nantes
April 30, 2024

Aim of the talk

Introduce a formalism to optimize motions of crowds by a central planner.

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- *central planner*: control on the dynamic of the ODE.
- *formalism*: Hamilton-Jacobi-Bellman equations.

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Figure: Invitation to the geometry coming from optimal transport!

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Notations

Throughout the talk,

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- $\#$ is the pushforward operator: given X, Y are topological spaces, $f : X \to Y$ measurable and $\mu \in \mathscr{P}(X)$, the measure $f\#\mu \in \mathscr{P}(Y)$ is defined by

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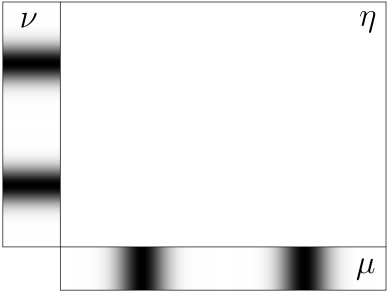
Example If $X = \mathsf{T}\mathbb{R}^d$, $Y = \mathbb{R}^d$ and $\pi_x : (x, v) \to x$ is the projection on the point coordinate, then any $\xi \in \mathscr{P}(\mathsf{T}\mathbb{R}^d)$ has for *base measure* $\pi_x\#\xi \in \mathscr{P}(\mathbb{R}^d)$.

The Wasserstein distance between measures

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures,



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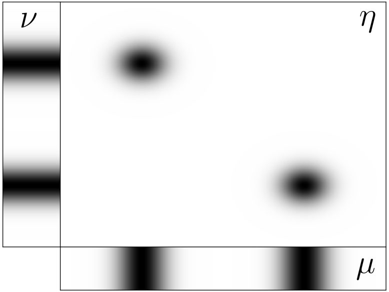
$$\Gamma(\mu, \nu) := \left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \pi_x \# \eta = \mu,\ \pi_y \# \eta = \nu \right\}.$$



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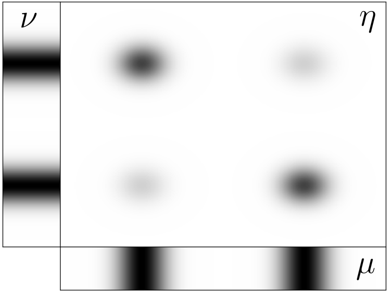

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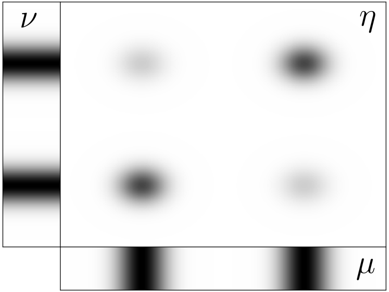

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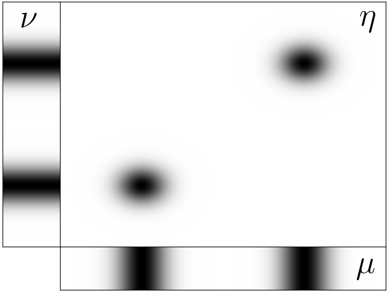

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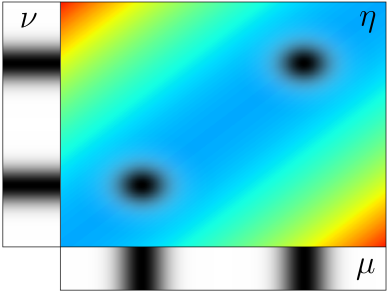
The Monge-Kantorovich distance with order $p \in [1, \infty)$ is

$$d^p_{\mathcal{W},p}(\mu,\nu) := \inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^p \, d\eta(x,y).$$



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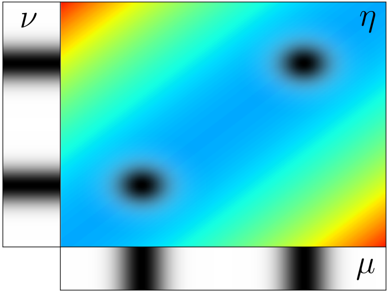

Def We call **Wasserstein space** the set $\mathscr{P}_2(\mathbb{R}^d) := \{\mu \in \mathscr{P}(\mathbb{R}^d) \mid d_{\mathcal{W}}(\mu,\delta_0) < \infty\}$, endowed with the distance $d_{\mathcal{W}} = d_{\mathcal{W},2}$ associated with the quadratic cost $|x-y|^2$.

Interpretation of the Wasserstein distance

The topology induced by $d_{\mathcal{W}}$

- is weaker than that induced by the total variation $|\mu|_{\mathsf{TV}} = \sup_{\mathcal{P}} \sum_{P \in \mathcal{P}} |\mu(P)|$, where $\mathcal{P}$ ranges in countable Borel partitions. For instance, $|\delta_x - \delta_y|_{\mathsf{TV}} = 2$ whenever $x \neq y$, but $d_{\mathcal{W}}(\delta_x, \delta_y) = |x - y|$.

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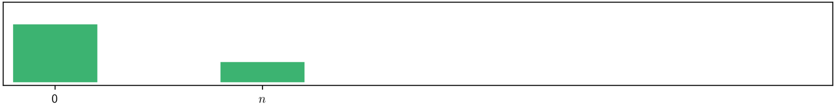

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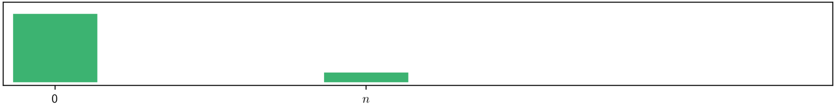

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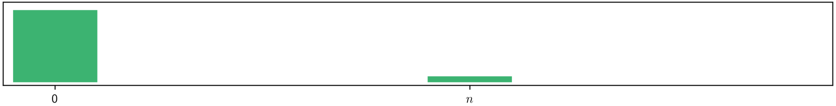

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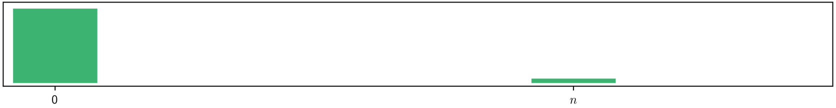

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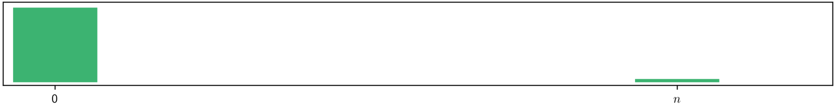

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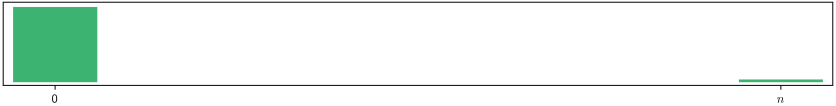

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  Duality with continuous functions having quadratic growth (instead of continuous and bounded functions).
- does not induce convergence of supports: the centered Gaussian with variance ε converges (in Wasserstein) towards δ_0 when $\varepsilon \to 0$, but has full support for all $\varepsilon > 0$.

Absolutely continuous curves

A curve $(y_t)_{t\in[0,1]}$ is called $2-$absolutely continuous if there exists $g \in L^2(0,1;\mathbb{R}^+)$ such that

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In $\mathbb{R}^d$, equivalently "curves that are integrals of their derivative". Similar result in $\mathscr{P}_2(\mathbb{R}^d)$:

Theorem – Characterization of AC curves [AGS05] A curve of measures $(\mu_t)_{t\in[0,1]}$ is absolutely continuous in $\left(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}}\right)$ if and only if there exists an a.e.-defined measurable curve $(v_t)_{t\in[0,1]}$, with $v_t \in L^2_{\mu_t}(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$ for a.e. $t \in [0,1]$, such that

$$\partial_t \mu_t + \text{div}\,(v_t \# \mu_t) = 0$$

in the sense of distributions.

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Theorem – Bonnet-Frankowska 2023 Assume that $f : [0,T]\times\mathbb{R}^d\times\mathscr{P}_2(\mathbb{R}^d) \to \mathrm{T}\mathbb{R}^d$ is measurable in time, continuous in space and measure. Then from any $\mu_0 \in \mathscr{P}_2(\mathbb{R}^d)$ is issued at least one solution of ($\mathrm{Cont}^{\mathrm{y}}$). If moreover f is Lipschitz-continuous in space and measure, uniqueness and estimates.

Representation

Example of admissible dynamic:

$$f(t,x,\mu) := g(t,x) + \int_{y\in\mathbb{R}^d} \psi(y)d\mu(y),$$

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It can be shown that any solution $(\mu_t)_{t\in[0,T]}$ of the continuity equation (Cont$^{\mathsf{y}}$) is representable as a superposition of *flow lines*: there exists a measure $\eta \in \mathscr{P}_2(\mathsf{AC}([0,T];\mathbb{R}^d))$ such that

$$\mu_t = e_t \# \eta, \qquad \text{i.e.} \qquad \int_{x\in\mathbb{R}^d} \varphi(x)d\mu_t(x) = \int_{\gamma\in\mathsf{AC}([0,T];\mathbb{R}^d)} \varphi(\gamma_t)d\eta(\gamma),$$

and η is concentrated on the solutions of the ODE system $\dot{\gamma}_t = f(t,\gamma_t,\mu_t)$.

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$$\mathfrak{J}(y_T^{0,x,u}) \leqslant \mathfrak{J}(y_T^{0,x,v}),$$

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- Contains formulations with running cost and/or optimal stopping time problem.
- Pontryagin maximum principle, Ricatti equation (linear quadratic case), Bellman principle.

Why formulate it with measures?

Taking $\mathscr{P}_2(\mathbb{R}^d)$ as the state space arise naturally from

- crowd motion: individuals (sum of Dirac masses) or crowds (measures with density), controlled by a central planner (flock of drones). *Not mean field games.*

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- crowd motion: individuals (sum of Dirac masses) or crowds (measures with density), controlled by a central planner (flock of drones). *Not mean field games.*
- robust control and/or physical uncertainty: consider not only the trajectory of one point, but also of a distribution around neighbours.

Solving control problems via dynamic programming

Idea of dynamic programming:

- introduce the *value function* $V(t,x) := \inf_{u \in L^\infty(t,T;U)} \mathfrak{J}(y_T^{t,x,u})$.

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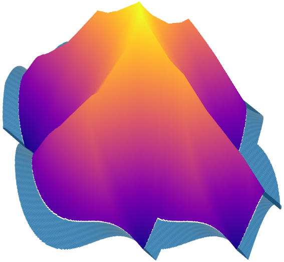

Problem: the value function is usually not smooth.

Typical example: distance to the boundary.

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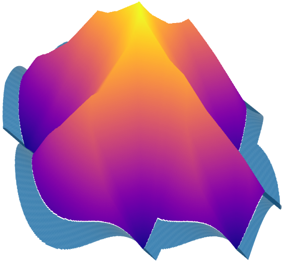

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Typical example: distance to the boundary. Thus need for

- nonsmooth analysis to recover the optimal trajectory,

Solving control problems via dynamic programming

Idea of dynamic programming:

- introduce the *value function* $V(t,x) := \inf_{u \in L^\infty(t,T;U)} \mathfrak{J}(y_T^{t,x,u})$.
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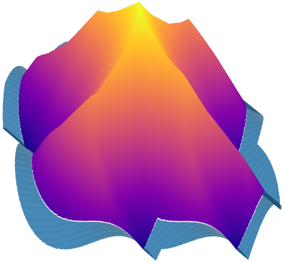


Problem: the value function is usually not smooth.

Typical example: distance to the boundary. Thus need for

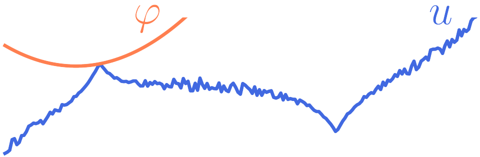
- nonsmooth analysis to recover the optimal trajectory,
- viscosity solutions to give a meaning to the PDE.

Viscosity solutions in short

Consider $H(x, \nabla_x u(x)) = 0$ in an open $\Omega \subset \mathbb{R}^d$. Viscosity solutions are equivalently defined by

- smooth test functions:

u is a **subsolution** if it is **u.s.c**, satisfies $u \leqslant \mathfrak{J}$ on $\partial\Omega$, and if whenever $\varphi \in \mathcal{C}^1$ is such that $u - \varphi$ reaches a **maximum** at x,



there holds $H\left(x, \nabla\varphi(x)\right) \leqslant 0$.

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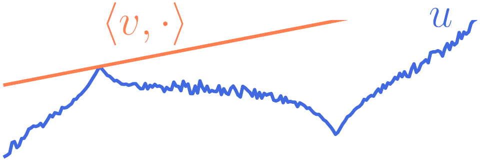

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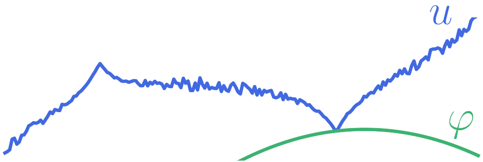

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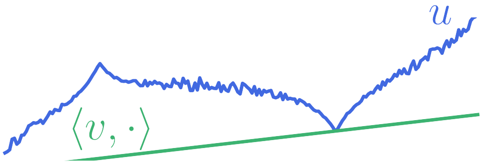

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Ishii 1985: discontinuous solut°s

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- Perron's method for proving **existence** by contradiction, using uniqueness and stability.

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The problem

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The Wasserstein space is attractive for its structure: metric, geodesic, curved space, good notions of trajectories, easy interpretation, setting of mean field games, applications...

But how to understand differential calculus?

The Lions formulation (1/2)

Back to the beginning: *differential calculus* should offer tools to characterize and control the *local variations of a function*. **Variations along which curves?**

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Proposition – Moving around regular measures [BBG02] If $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ is "sufficiently regular" (thinks absolutely continuous w.r.t. the Lebesgue measure), then any geodesic (in the Wasserstein space) leaving μ is of the form

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In a sense, *linearization* of $\mathscr{P}_2(\mathbb{R}^d)$ around μ by using the Hilbert space $L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$.

The Lions formulation (2/2)

Def – Lions-Gangbo derivative An application $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is differentiable in the Lions sense at $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ if there exists an element $f \in L^2_\mu(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$ such that

$$u(\nu) - u(\mu) = \int_{(x,y)\in(\mathbb{R}^d)^2} \langle f(x), y - x \rangle \, d\eta(x,y) + o(d_{\mathcal{W}}(\mu,\nu))$$

for any $\eta \in \Gamma_o(\mu,\nu)$ an optimal transport plan, and any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$.

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- Then $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $f : \mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d) \to \mathrm{T}\mathbb{R}^d$ such that for each μ, the element $f(\cdot,\mu)$ belongs to L^2_μ and is a gradient of u.

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- Provides a set of admissible *test functions* to define viscosity solutions!

The good and the bad about this definition

- Extensive litterature, various equivalent formulations of the definition, growing corpus of results of existence, uniqueness and stability of smooth solutions for mean field games systems [CD18, CDLL19, DS23].

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- The “linearization” of the space by L^2_μ is not valid whenever μ is degenerated: for instance, if $\mu = \delta_0$, only allows to move towards other Dirac masses.
- The theory of viscosity solutions is *global* and *uses every point*. Inconsistencies appear when trying to link the viscosity theory using Lions-Gangbo derivatives, and metric viscosity [AF14].

Directional derivatives

Back again to the beginning: variations along elements of the (larger) *geometric tangent space* $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \subset \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$ [Gig08].

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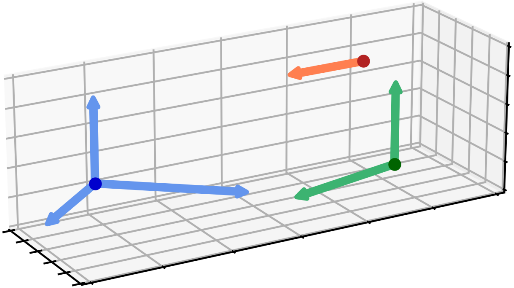

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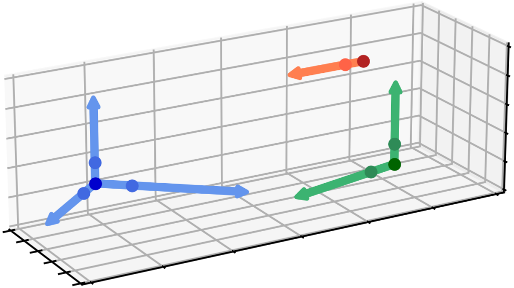

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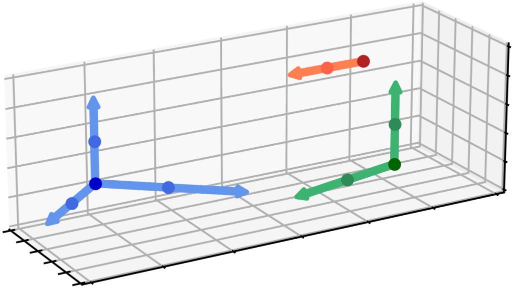

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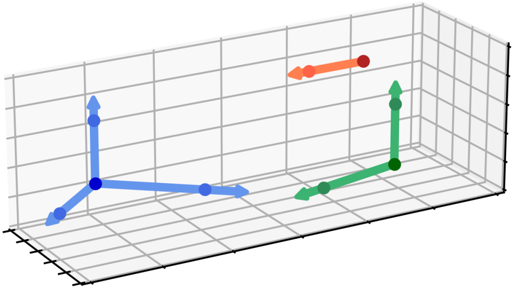

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Def – Directional derivatives The *differential* of an application $u : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ at a measure μ is the application

$$D_\mu u : \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}, \qquad D_\mu u(\xi) := \lim_{h \searrow 0} \frac{u\left((\pi_x + h\pi_v)\#\xi\right) - u(\mu)}{h}.$$

Viscosity solutions with directional derivatives

Let H be defined on couples (μ, p), where $p : \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is sufficiently regular (for instance, Lipschitz-continuous, positively homogeneous, representable by inf/sup over some set...). Consider the equation

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- Inspired from the theory developed in non-positively curved spaces by O. Jerhaoui [Jer22].

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Strong comparison principle

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Theorem – Comparison principle [JPZ23] Assume that H is positively homogeneous and Lipschitz-continuous in its second argument, and satisfies

$$H\left(\mu, -D_\mu d_{\mathcal{W}}^2(\cdot,\nu)\right) - H\left(\nu, D_\nu d_{\mathcal{W}}^2(\mu,\cdot)\right) \leqslant C d_{\mathcal{W}}^2(\mu,\nu)$$

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for some constant $C \geqslant 0$. Then for any bounded subsolution $u : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ and bounded supersolution $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$, there holds

$$\sup_{(t,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)} u(t,\mu) - v(t,\mu) \leqslant \sup_{\mu\in\mathscr{P}_2(\mathbb{R}^d)} u(T,\mu) - v(T,\nu).$$

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- A version exists in compact manifolds without borders [Jer22].
- Assumptions on H typical for control problems, not satisfied for game problems.

Theorem – Characterization of the value function Assume that the dynamic of the Mayer problem is bounded and Lipschitz-continuous in all its arguments, and that $\mathfrak{J}$ is Lipschitz-continuous. Then the value function is the unique viscosity solution of the HJB equation

$$-\partial_t V(t,\mu) + \sup_{u\in U} -D_\mu V\left(f(\cdot,\mu)\#\mu\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu).$$

The case of possibly infinite cost

Assume now that $\mathfrak{J} : \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ is *weak-$*$ lower semicontinuous on every Wasserstein ball*, and that the dynamic does not depend on the measure variable.

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- Uses a (nice) topology rendering $\mathscr{P}_2(\mathbb{R}^d)$ locally compact, removes a lot of technicalities.
- Proceeds by approximation of $\mathfrak{J}$ from below, and uses well-posedness in this case.

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Challenges

- Purely first order techniques, seems difficult to adapt to second-order.

Strengths and weaknesses

Advantages

- No need to restrict over a smooth dense set, or to rely on ellipticity.
- Quite simple formulation, coherent across a large family of geodesic spaces.
- When applicable, coherent with metric viscosity.

Challenges

- Purely first order techniques, seems difficult to adapt to second-order.
- Depending on the curvature of the space, lack of results concerning stability. Typically, no result in the Wasserstein space.

Conclusion

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Perspectives

- Include state constraints (ongoing work of E. Treumún).
- Other base space than $\mathbb{R}^d$ (ongoing work).

Thank you!

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