Some recent developments in Wasserstein geometry

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The metric space $(\mathscr{P}_2, W)$

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Reached for $\eta \in \Gamma_o(\mu, \nu)$.

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A curve $\gamma : [0,1] \to \mathscr{P}_2(\mathbb{R}^d)$ is a geodesic if

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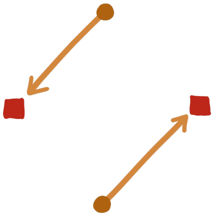

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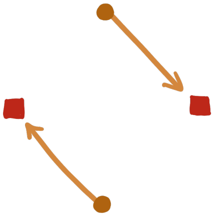

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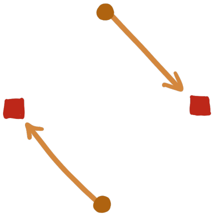

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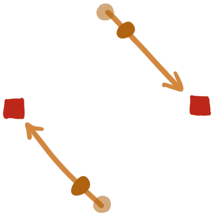

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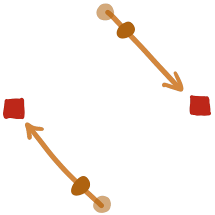

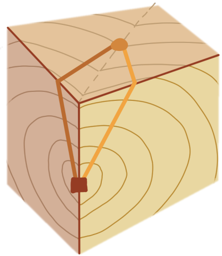

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Along any geodesic γ, there holds

$$W^2(\mu,\gamma_t) \geqslant (1-t)W^2(\mu,\gamma_0) + tW^2(\mu,\gamma_1) - t(1-t)W^2(\gamma_0,\gamma_1).$$

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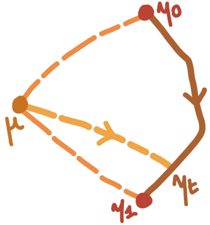

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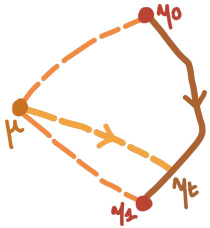

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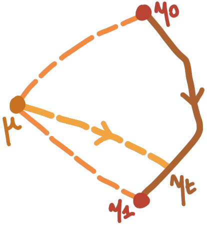


Same 2-concavity as on a corner. Following the analogy, we can compute $W^2(\mu,\nu)$ for ν in the red segment in Fig 1, i.e. from ν_0 to ν_1 in Fig 2.

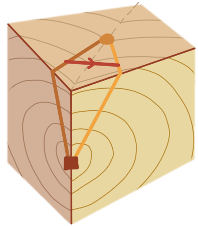


Fig. 1

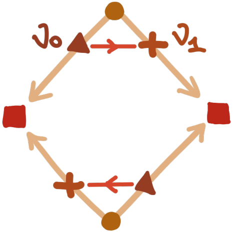
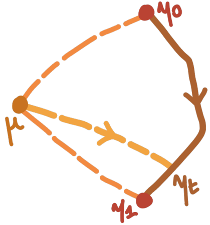


Fig. 2

Curved space

Along any geodesic γ, there holds

$$W^2(\mu,\gamma_t) \geqslant (1-t)W^2(\mu,\gamma_0) + tW^2(\mu,\gamma_1) - t(1-t)W^2(\gamma_0,\gamma_1).$$



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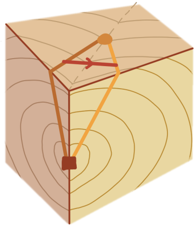

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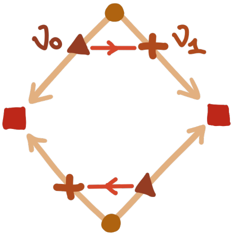

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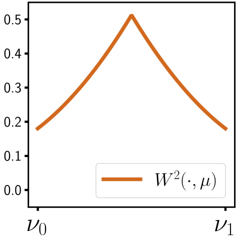

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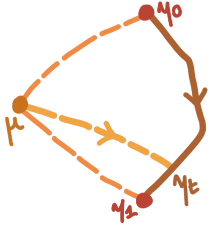

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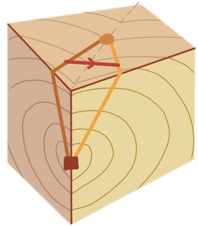

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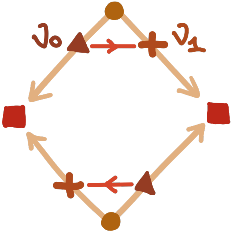

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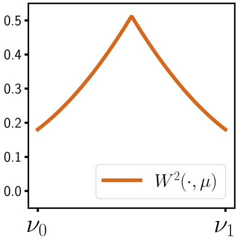

Fruitful intuition: curvature comes from identifying particles forming the measure.

Dual problem (1/2)

Computing $W^2(\mu,\nu)$ amounts to optimize under the constraints $\pi_{x\#}\eta = \mu$ and $\pi_{y\#}\eta = \nu$.

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$$= \sup_{\varphi,\psi} \int \psi d\nu - \int \varphi d\mu,$$

where the last supremum runs on φ, ψ such that $|x-y|^2 + \varphi(x) - \psi(y) \geqslant 0$.

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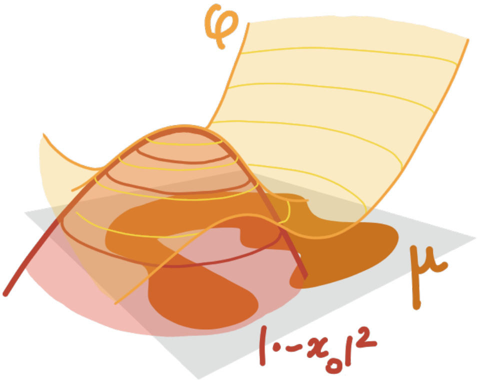

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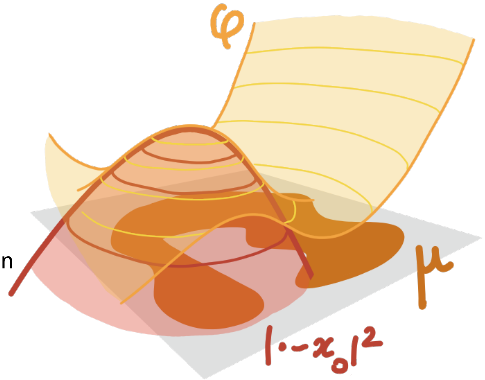

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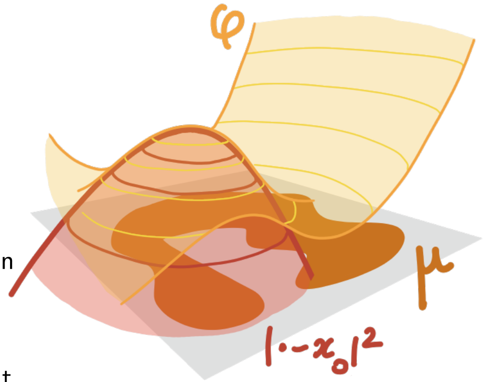

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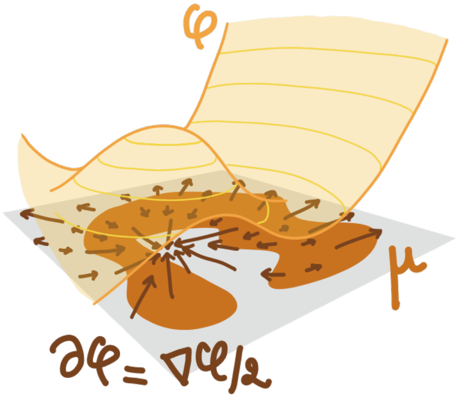

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Two measures μ, ν are in convex order if there exists $\eta \in \Gamma(\mu, \nu)$ such that

$$\eta = \mu \otimes \eta_x, \qquad \int_{y \in \mathbb{R}^d} y \, d\eta_x(y) = x \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

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Mainly studied for applications in finance, evaluation of martingale processes (Backhoff-Veraguas, Beiglböck, Pammer, Juillet, Schachermayer, Acciaio, Loeper...).

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Definition Given $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, let

$$\mathcal{C}_\mu^+ := \left\{\nu \in \mathscr{P}_2(\mathbb{R}^d) \;\middle|\; \mu \leqslant_c \nu\right\}, \qquad \mathcal{C}_\mu^- := \left\{\nu \in \mathscr{P}_2(\mathbb{R}^d) \;\middle|\; \nu \leqslant_c \mu\right\}.$$

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Consequently, any $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ admits a unique metric projection p_ν on $\mathcal{C}_\mu^-$, and the unique optimal transport plan between ν and p_ν is of the form $\eta = (id, T)_\# \nu$ for some $T \in L^2_\nu$.

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Theorem – [ACJ20][1] For $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$, there holds

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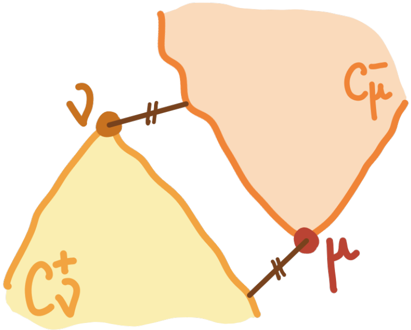

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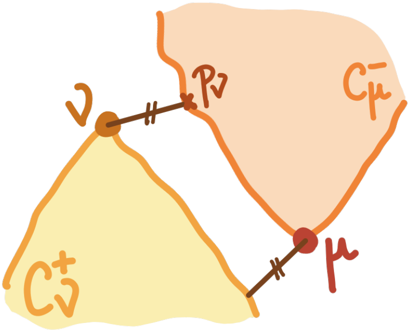

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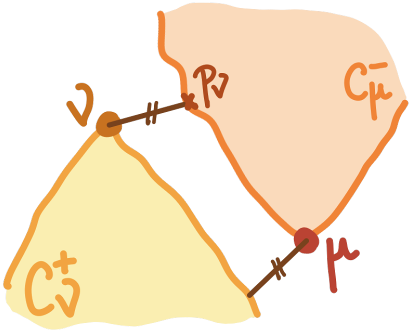

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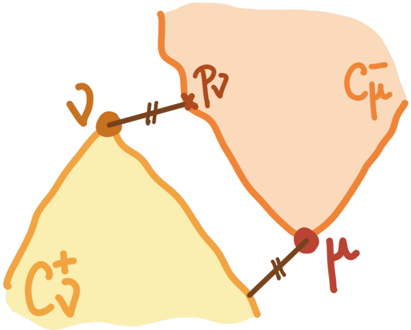

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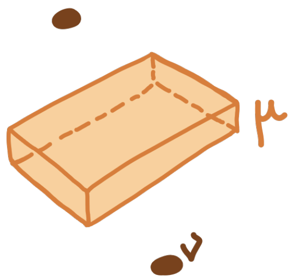

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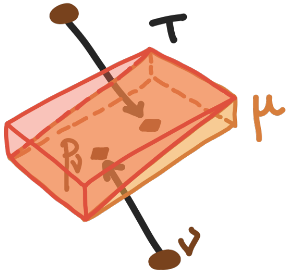

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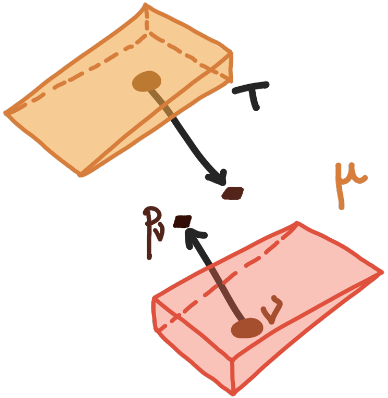

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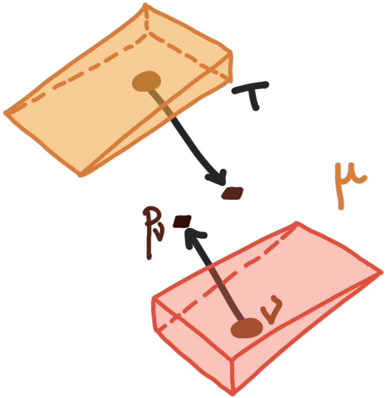

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Then $\omega \in \mathcal{C}_\nu^+$, and $W^2(\mu, \omega) = W^2\left(\mu, \mathcal{C}_\nu^+\right)$. Application in sampling in convex order [Kim+25][2].

[1] A. Alfonsi, J. Corbetta, and B. Jourdain, "Sampling of probability measures in the convex order by Wasserstein projection" (2020).
[2] J. Kim, Y.-H. Kim, Y. Ruan, and A. Warren, *Statistical inference of convex order by Wasserstein projection* (2025).

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This "backward" decomposition has a "forward" analogue [KR21][2].

[1] N. Gozlan and N. Juillet, "On a mixture of Brenier and Strassen Theorems" (2020).
[2] Y.-H. Kim and Y. L. Ruan, *Backward and Forward Wasserstein Projections in Stochastic Order* (2021).

Extension of geodesics

Let η induce a geodesic γ. One could naïvely declare

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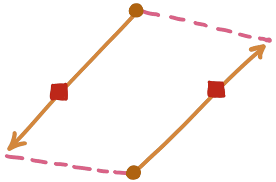

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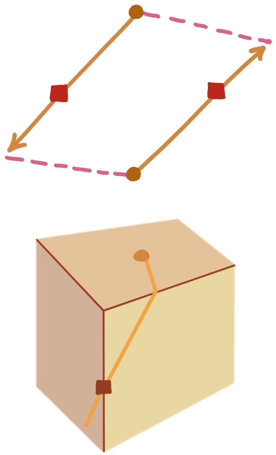

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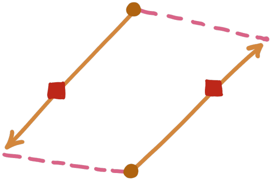

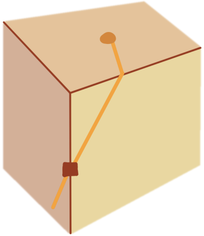

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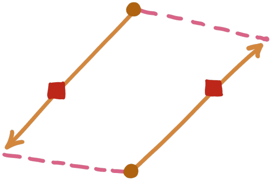

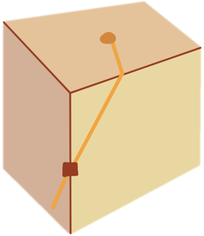

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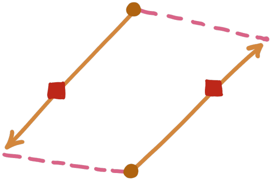

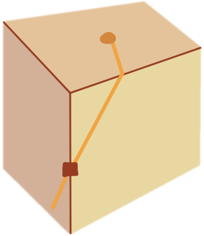

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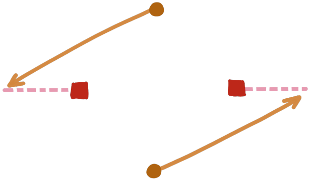

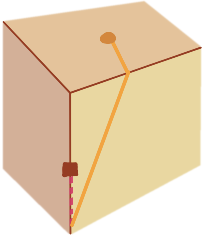

[1] T. Gallouët, A. Natale, and G. Todeschi, "From geodesic extrapolation to a variational BDF2 scheme for Wasserstein gradient flows" (2022). See also T. O. Gallouët, A. Natale, and G. Todeschi, *Metric extrapolation in the Wasserstein space* (2025).

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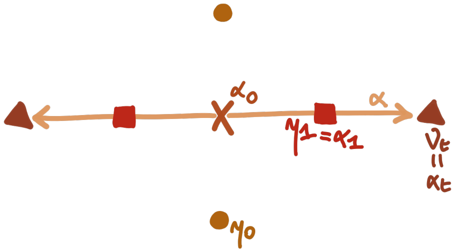

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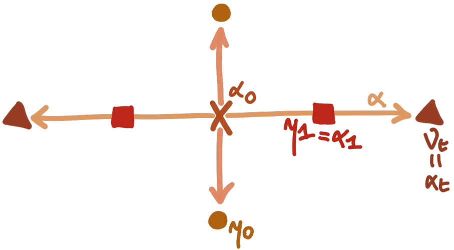

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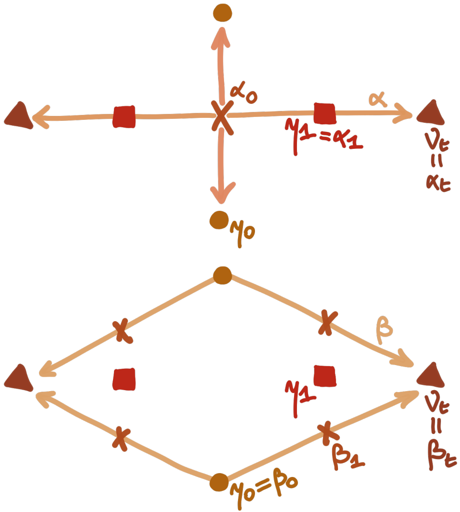

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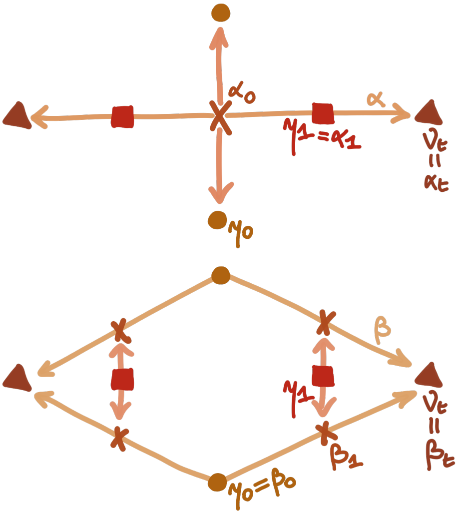

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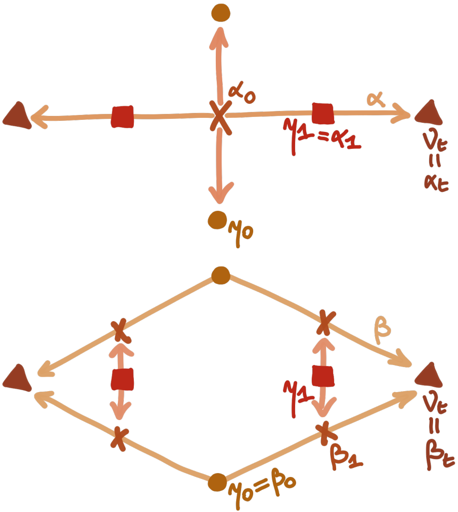
Arises from (many) dual formulations of the problem (P_t) (again Gallouët-Natale-Todeschi).



Regularization of W^2

Almost simultaneously, [BL24][1] introduced the sup-convolution (with $\lambda > 1$ large)

$$\Phi_\lambda(\mu, \mu_0) := \sup_{\nu \in \mathscr{P}_2(\mathbb{R}^d)} W^2(\mu_0, \nu) - \lambda W^2(\mu, \nu).$$

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Theorem The function $\Phi_\lambda(\cdot,\mu_0)$ is C^1 in the sense of Lions. As $\lambda\to\infty$, its Wasserstein gradient at μ converges in L^2_μ to the element of minimal norm of the superdifferential of $W^2(\cdot,\mu_0)$.

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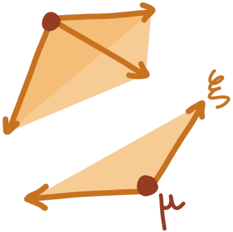
A first orthogonal decomposition

Let $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$.



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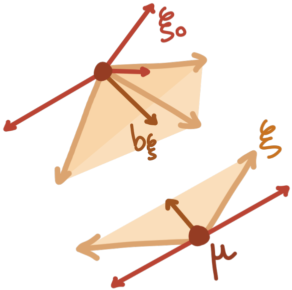

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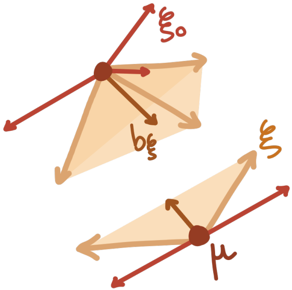

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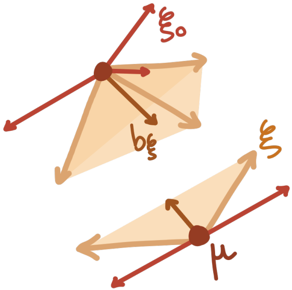

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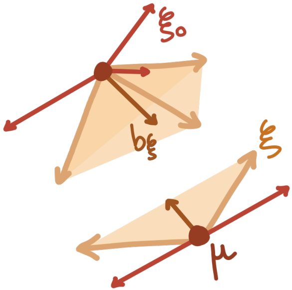

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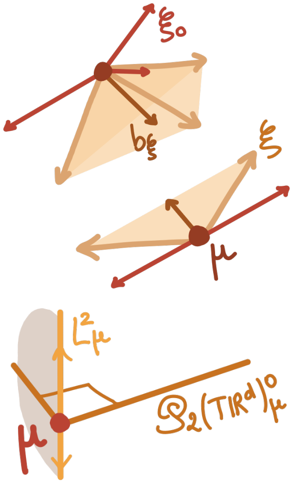

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In general, any $\xi \in \mathbf{Tan}_\mu$ is a superposition of elements of Tan_μ, but the converse is false.

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Centred factor of $\mathbf{Tan}_\mu$

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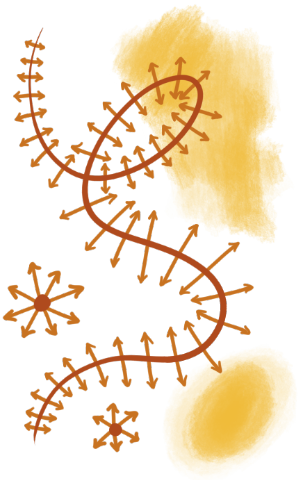

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Generalizes [Lot16][2] to any measure.

[1] A. Aussedat, *Locality of centred tangent cones in the Wasserstein space* (2025).
[2] J. Lott, "On tangent cones in Wasserstein space" (2016).

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The definition of $\xi \in \mathbf{Tan}_\mu$ is equivalent to being a W_μ-limit point of a sequence $(\xi_n)_n$ such that

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[1] A. Aussedat, "Optimal control problems and Hamilton-Jacobi-Bellman equations in some curved metric spaces" (2025).

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$$\frac{W(\mu, (\pi_x + h\pi_v)_\# \xi_n)}{h} \equiv \|\xi_n\|_\mu := \left(\int |v|^2 d\xi_n \right)^{1/2} \qquad \forall\, h \in (0, h_n].$$

Theorem – [Aus25b][1] Assume that $\xi = (id, f)_\# \mu$ for some $f \in L^2_\mu(\mathbb{R}^d; \mathbb{R}^d)$. Then

$$\xi \in \mathbf{Tan}_\mu \qquad \Longleftrightarrow \qquad \lim_{h \searrow 0} \frac{W(\mu, (id + hf)_\# \mu)}{h} = \|\xi\|_\mu = \|f\|_{L^2_\mu}.$$

[1] A. Aussedat, "Optimal control problems and Hamilton-Jacobi-Bellman equations in some curved metric spaces" (2025).

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$d > 1$; both (1) and (2) are open in the general case.

The first problem in dimension one

A set $A \subset \mathbb{R}$ belongs to $\mathcal{A}$ if there exists vanishing sequences $(s_n)_n$ and $(\tau_n)_n$ such that for any $x \in A$,

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Theorem – [Aus26][1] Let $\mu \in \mathscr{P}_2(\mathbb{R})$. The following are equivalent:

- $\mathbf{Tan}_\mu$ coincides with the set of $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ inducing a curve of initial velocity $\|\xi\|_\mu$;

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Le reste, je ne le sais pas.
(Beyond, I don't know.)

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Table of Contents

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The tangent cone $\mathcal{T}_\mu$ is given by the elements $\sigma \in L^1_\mu(\mathbb{R}^d; \mathbb{R}^d)$ such that $-\operatorname{div}(\sigma\mu) \in \text{Æ}$.

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Properties

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Thank you for your attention!