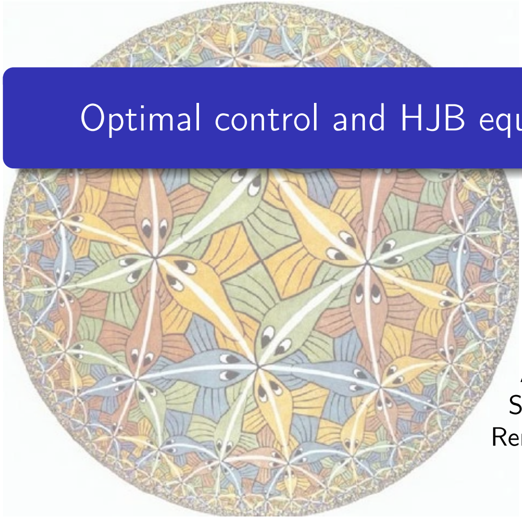
Optimal control and HJB equations in non-positively curved spaces



Averil Aussedat
Joint work with Othmane Jerhaoui and Hasnaa Zidani

April 2, 2026
Séminaire d'analyse numérique de l'IRMAR
Rennes

INSTITUT NATIONAL DES SCIENCES APPLIQUÉES ROUEN

LMI - EA 3226

Aim

Solve optimal control problems on metric spaces that are not smooth.

Aim

Solve optimal control problems on metric spaces that are not smooth.
Precisely,

- get a formulation of ODEs that would allow for Cauchy-Lipschitz theorems;

Aim

Solve optimal control problems on metric spaces that are not smooth.
Precisely,

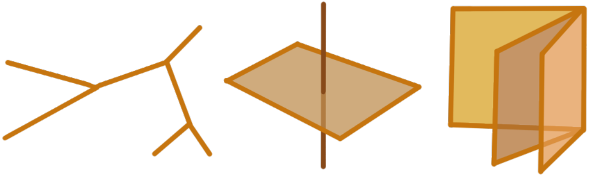
- get a formulation of ODEs that would allow for Cauchy-Lipschitz theorems;
- see whether the associated control problems can be treated with the (previously existing) theory of Hamilton-Jacobi equations in these spaces.

Table of Contents

Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

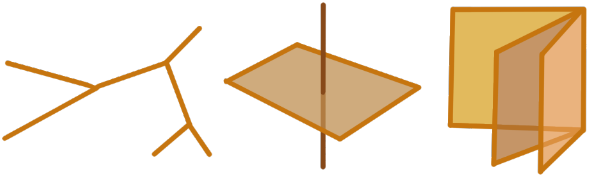
- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures



Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures

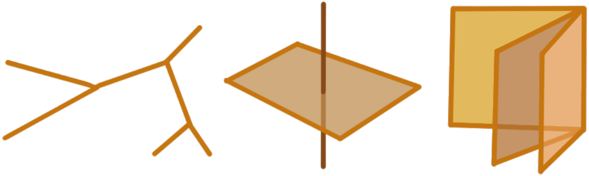


All these models are gluings of Euclidean spaces, and blow-ups of potentially more complicated spaces.

Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures



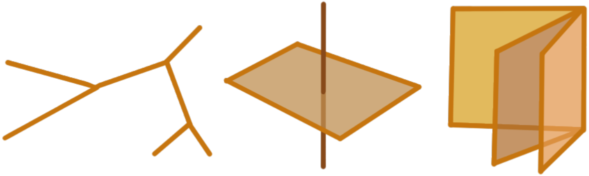
All these models are gluings of Euclidean spaces, and blow-ups of potentially more complicated spaces. Several works (and the ANR COSS!) to understand HJ PDEs on them (not exhaustive list):

- Achdou, Camilli, Cutrí & Tchou 2011-2013, introducing penalized formulations. Then Camilli, Schieborn & Marchi 2013, recent works of Achdou on homogenization.

Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures



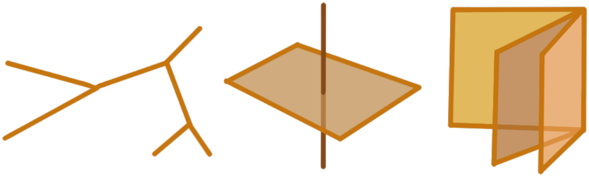
All these models are gluings of Euclidean spaces, and blow-ups of potentially more complicated spaces.
Several works (and the ANR COSS!) to understand HJ PDEs on them (not exhaustive list):

- Achdou, Camilli, Cutrí & Tchou 2011-2013, introducing penalized formulations. Then Camilli, Schieborn & Marchi 2013, recent works of Achdou on homogenization.
- Barles, Briani & Chasseigne 2013-2014, Rao & Zidani 2013, Barnard-Wolenski 2013 on stratified formulations (recent big book of Barles and Chasseigne).

Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures



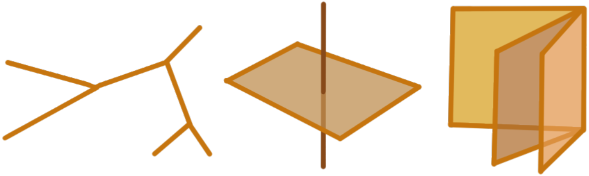
All these models are gluings of Euclidean spaces, and blow-ups of potentially more complicated spaces. Several works (and the ANR COSS!) to understand HJ PDEs on them (not exhaustive list):

- Achdou, Camilli, Cutrí & Tchou 2011-2013, introducing penalized formulations. Then Camilli, Schieborn & Marchi 2013, recent works of Achdou on homogenization.
- Barles, Briani & Chasseigne 2013-2014, Rao & Zidani 2013, Barnard-Wolenski 2013 on stratified formulations (recent big book of Barles and Chasseigne).
- Link with discontinuities in Imbert, Monneau & Zidani 2013, then Imbert & Monneau 2017.

Motivation

Problems posed on spaces with junctions, or abrupt change of dimension.

- 1D networks without loops
- layers connected by beams
- books, higher-dimensional structures



All these models are gluings of Euclidean spaces, and blow-ups of potentially more complicated spaces. Several works (and the ANR COSS!) to understand HJ PDEs on them (not exhaustive list):

- Achdou, Camilli, Cutrí & Tchou 2011-2013, introducing penalized formulations. Then Camilli, Schieborn & Marchi 2013, recent works of Achdou on homogenization.
- Barles, Briani & Chasseigne 2013-2014, Rao & Zidani 2013, Barnard-Wolenski 2013 on stratified formulations (recent big book of Barles and Chasseigne).
- Link with discontinuities in Imbert, Monneau & Zidani 2013, then Imbert & Monneau 2017.

Observation of O. Jerhaoui: gluings are instances of *CAT(0) spaces*.

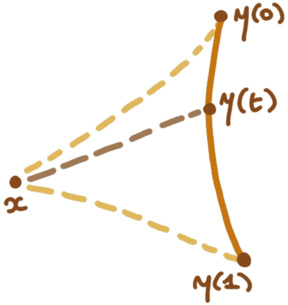
CAT(0) spaces

Polish spaces which are *geodesic* (any two points are linked by at least one geodesic)

CAT(0) spaces

Polish spaces which are *geodesic* (any two points are linked by at least one geodesic) and *non-positively curved:*

Definition If γ is a geodesic and $x \in \Omega$, the distance $d(x, \gamma(t))$ is smaller than what it would be in the plane.



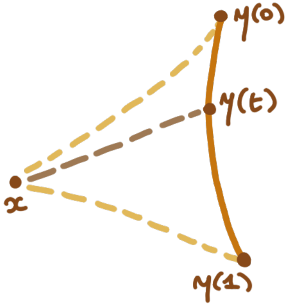
CAT(0) spaces

Polish spaces which are *geodesic* (any two points are linked by at least one geodesic) and *non-positively curved:*

Definition If γ is a geodesic and $x \in \Omega$, the distance $d(x, \gamma(t))$ is smaller than what it would be in the plane.

Quantitatively,

$$d^2(x, \gamma_t) \leqslant (1-t)d^2(x, \gamma_0) + td^2(x, \gamma_1) - t(1-t)d^2(\gamma_0, \gamma_1).$$



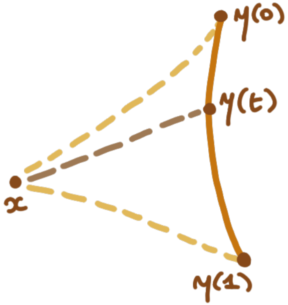
CAT(0) spaces

Polish spaces which are *geodesic* (any two points are linked by at least one geodesic) and *non-positively curved:*

Definition If γ is a geodesic and $x \in \Omega$, the distance $d(x, \gamma(t))$ is smaller than what it would be in the plane.

Quantitatively,

$$d^2(x, \gamma_t) \leqslant (1-t)d^2(x, \gamma_0) + td^2(x, \gamma_1) - t(1-t)d^2(\gamma_0, \gamma_1).$$

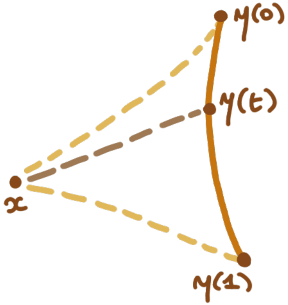


- Equivalently, $d^2(x, \cdot)$ is 2-semiconvex along geodesics. Hence uniqueness of geodesics.

CAT(0) spaces

Polish spaces which are *geodesic* (any two points are linked by at least one geodesic) and *non-positively curved:*

Definition If γ is a geodesic and $x \in \Omega$, the distance $d(x, \gamma(t))$ is smaller than what it would be in the plane.



Quantitatively,

$$d^2\left(x, \gamma_t\right) \leqslant (1-t)d^2(x, \gamma_0) + td^2(x, \gamma_1) - t(1-t)d^2(\gamma_0, \gamma_1).$$

- Equivalently, $d^2(x, \cdot)$ is 2-semiconvex along geodesics. Hence uniqueness of geodesics.
- Example which is *not* so: the sphere. ⚠ At best local models (for traffic networks, for instance).

Canonical construction of tangent cones

Manipulating the squared distance inequality, we get that for two geodesics γ, γ' issued from the same x,

$$t \mapsto \frac{d(\gamma(t), \gamma'(t))}{t} \qquad \text{is nondecreasing.}$$

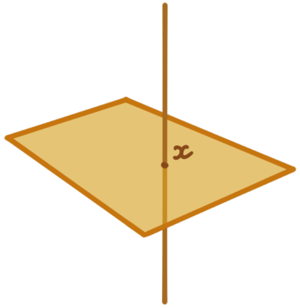
Canonical construction of tangent cones

Manipulating the squared distance inequality, we get that for two geodesics γ, γ' issued from the same x,

$$t \mapsto \frac{d(\gamma(t), \gamma'(t))}{t} \qquad \text{is nondecreasing.}$$

Definition – Tangent cone Let $\mathrm{T}_x\Omega$ be the set of limit points of reparametrized geodesics, with respect to

$$d_x(\gamma, \gamma') := \lim_{t \searrow 0} \frac{d(\gamma(t), \gamma'(t))}{t}.$$



Canonical construction of tangent cones

Manipulating the squared distance inequality, we get that for two geodesics γ, γ' issued from the same x,

$$t \mapsto \frac{d(\gamma(t), \gamma'(t))}{t} \qquad \text{is nondecreasing.}$$

Definition – Tangent cone Let $\mathrm{T}_x\Omega$ be the set of limit points of reparametrized geodesics, with respect to

$$d_x(\gamma, \gamma') := \lim_{t \searrow 0} \frac{d(\gamma(t), \gamma'(t))}{t}.$$

- Systematic definition, extends the intuition on gluings.

Canonical construction of tangent cones

Manipulating the squared distance inequality, we get that for two geodesics γ, γ' issued from the same x,

$$t \mapsto \frac{d(\gamma(t), \gamma'(t))}{t} \qquad \text{is nondecreasing.}$$

Definition – Tangent cone Let $\mathrm{T}_x\Omega$ be the set of limit points of reparametrized geodesics, with respect to

$$d_x(\gamma, \gamma') := \lim_{t \searrow 0} \frac{d(\gamma(t), \gamma'(t))}{t}.$$

- Systematic definition, extends the intuition on gluings.
- Provides curves along which one can differentiate.

Table of Contents

Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.

Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.
- Appropriate definition by "nonlinear testing":

Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.
- Appropriate definition by "nonlinear testing":

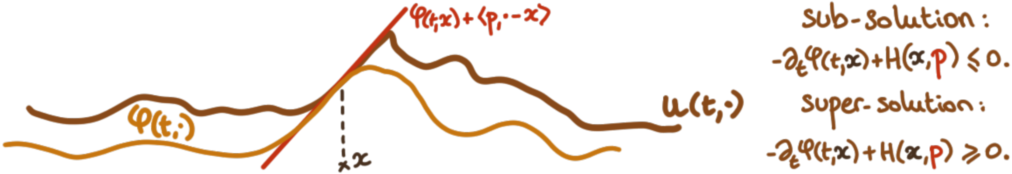
Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.
- Appropriate definition by "nonlinear testing":



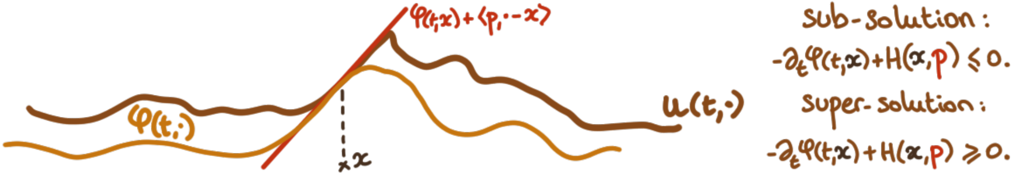
Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.
- Appropriate definition by "nonlinear testing":



- Smoothness of test functions gives stability.

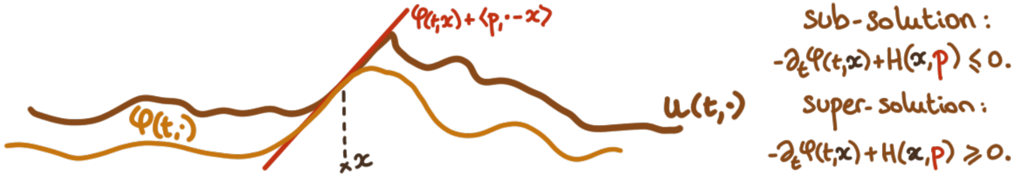
Model problem

The starting point of the literature is usually equation of the form

$$-\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) = 0 \qquad + \text{terminal condition.}$$

Then more complicated questions of homogenization, viscous approximation using noise, etc.

- In $\mathbb{R}^d$, long history of solving these equations by *viscosity solutions*.
- Appropriate definition by "nonlinear testing":



- Smoothness of test functions gives stability. What if Ω does not support smooth functions?

What is done

Introduction of “partially smooth” test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each “good piece” is smooth.

What is done

Introduction of “partially smooth” test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each “good piece” is smooth.
- Camilli, Schieborn & Marchi 2013: different sets depending on input/output junctions.

What is done

Introduction of "partially smooth" test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each "good piece" is smooth.
- Camilli, Schieborn & Marchi 2013: different sets depending on input/output junctions.
- Rao & Zidani, Jerhaoui & Zidani: reducing to essential dynamics in stratified control problems.

What is done

Introduction of "partially smooth" test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each "good piece" is smooth.
- Camilli, Schieborn & Marchi 2013: different sets depending on input/output junctions.
- Rao & Zidani, Jerhaoui & Zidani: reducing to essential dynamics in stratified control problems.

Various purposes: discontinuity of the Hamiltonian, singular problems.

- In networks, involved construction by Imbert, Monneau & Zidani 2013, and Imbert & Monneau 2017, to transfer information from one branch to another.

What is done

Introduction of “partially smooth” test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each “good piece” is smooth.
- Camilli, Schieborn & Marchi 2013: different sets depending on input/output junctions.
- Rao & Zidani, Jerhaoui & Zidani: reducing to essential dynamics in stratified control problems.

Various purposes: discontinuity of the Hamiltonian, singular problems.

- In networks, involved construction by Imbert, Monneau & Zidani 2013, and Imbert & Monneau 2017, to transfer information from one branch to another.
- Flux limiters: conditions at the junction arising from limits of problems on thinning domains (Cardaliaguet, Briani-Ishii) or oscillating problems (Achdou, thesis of Ali Assi)

What is done

Introduction of “partially smooth” test functions.

- Achdou, Camilli, Cutrí & Tchou 2011-2013, Barles, Briani & Chasseigne 2013-2014: restriction to each “good piece” is smooth.
- Camilli, Schieborn & Marchi 2013: different sets depending on input/output junctions.
- Rao & Zidani, Jerhaoui & Zidani: reducing to essential dynamics in stratified control problems.

Various purposes: discontinuity of the Hamiltonian, singular problems.

- In networks, involved construction by Imbert, Monneau & Zidani 2013, and Imbert & Monneau 2017, to transfer information from one branch to another.
- Flux limiters: conditions at the junction arising from limits of problems on thinning domains (Cardaliaguet, Briani-Ishii) or oscillating problems (Achdou, thesis of Ali Assi)

Focus for the rest of the talk: continuous equations/Hamiltonians, setting of CAT(0) spaces.

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

$$\text{Take test functions that are } \begin{cases} \text{semiconvex} \\ \text{semiconcave} \end{cases} \text{ when touching from } \begin{cases} \text{above} \\ \text{below} \end{cases}.$$

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

$$\text{Take test functions that are } \begin{cases} \text{semiconvex} \\ \text{semiconcave} \end{cases} \text{ when touching from } \begin{cases} \text{above} \\ \text{below} \end{cases}.$$

Plenty of them: series and compositions of squared distances.

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

$$\boxed{\text{Take test functions that are } \begin{cases} \text{semiconvex} \\ \text{semiconcave} \end{cases} \text{ when touching from } \begin{cases} \text{above} \\ \text{below} \end{cases}.}$$

Plenty of them: series and compositions of squared distances. **Directionally differentiable!**

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

$$\text{Take test functions that are } \begin{cases} \text{semiconvex} \\ \text{semiconcave} \end{cases} \text{ when touching from } \begin{cases} \text{above} \\ \text{below} \end{cases}.$$

Plenty of them: series and compositions of squared distances. **Directionally differentiable!**

For the model equation $-\partial_t u + H(x, D_x u) = 0$,

- u subsolution if for any semiconvex φ touching u from above at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \leqslant 0$;

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

$$\text{Take test functions that are } \begin{cases} \text{semiconvex} \\ \text{semiconcave} \end{cases} \text{ when touching from } \begin{cases} \text{above} \\ \text{below} \end{cases}.$$

Plenty of them: series and compositions of squared distances. **Directionally differentiable!**

For the model equation $-\partial_t u + H(x, D_x u) = 0$,

- u subsolution if for any semiconvex φ touching u from above at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \leqslant 0$;
- u supersolution if for any semiconcave φ touching u from below at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \geqslant 0$.

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

> Take test functions that are $\begin{cases}\text{semiconvex} \\ \text{semiconcave}\end{cases}$ when touching from $\begin{cases}\text{above} \\ \text{below}\end{cases}$.

Plenty of them: series and compositions of squared distances. **Directionally differentiable!**

For the model equation $-\partial_t u + H(x, D_x u) = 0$,

- u subsolution if for any semiconvex φ touching u from above at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \leqslant 0$;
- u supersolution if for any semiconcave φ touching u from below at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \geqslant 0$.

> **Theorem – (Jerhaoui & Zidani 2023)** Comparison principle, stability, Perron's method.

Theory in CAT(0) spaces

From Jerhaoui & Zidani 2023, full details in the thesis of O. Jerhaoui:

> Take test functions that are $\begin{cases}\text{semiconvex}\\ \text{semiconcave}\end{cases}$ when touching from $\begin{cases}\text{above}\\ \text{below}\end{cases}$.

Plenty of them: series and compositions of squared distances. **Directionally differentiable!**

For the model equation $-\partial_t u + H(x, D_x u) = 0$,

- u subsolution if for any semiconvex φ touching u from above at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \leqslant 0$;
- u supersolution if for any semiconcave φ touching u from below at (t, x), $-\partial_t \varphi + H(x, D_x \varphi) \geqslant 0$.

> **Theorem – (Jerhaoui & Zidani 2023)** Comparison principle, stability, Perron's method.

Question now: can we apply it to control problems (which were the motivation...) ?

Table of Contents

Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \mathsf{curve}_{f(y_t)}(h))}{h} = 0.$$

Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \mathsf{curve}_{f(y_t)}(h))}{h} = 0.$$

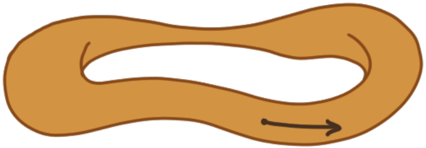
Problem: how to detect if f is Lipschitz? Not trivial to compare $f(x) \in \mathrm{T}_x\,\Omega$ and $f(y) \in \mathrm{T}_y\,\Omega$.

Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \text{curve}_{f(y_t)}(h))}{h} = 0.$$

Problem: how to detect if f is Lipschitz? Not trivial to compare $f(x) \in \mathrm{T}_x\,\Omega$ and $f(y) \in \mathrm{T}_y\,\Omega$.

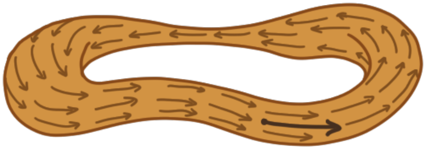


Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \mathsf{curve}_{f(y_t)}(h))}{h} = 0.$$

Problem: how to detect if f is Lipschitz? Not trivial to compare $f(x) \in \mathrm{T}_x\,\Omega$ and $f(y) \in \mathrm{T}_y\,\Omega$.

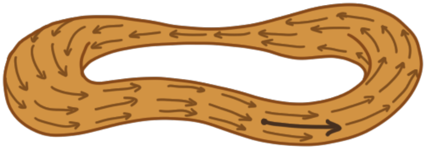


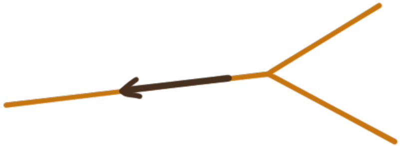
Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \mathsf{curve}_{f(y_t)}(h))}{h} = 0.$$

Problem: how to detect if f is Lipschitz? Not trivial to compare $f(x) \in \mathrm{T}_x\,\Omega$ and $f(y) \in \mathrm{T}_y\,\Omega$.



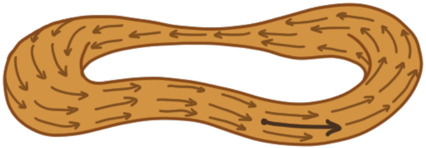


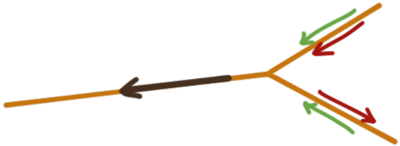
Definition of ODEs

Naive idea Since tangent cones are defined, let $f : \Omega \to \mathrm{T}\,\Omega$ be a section, and declare that $\dot{y}_t = f(y_t)$ if for a.e. t,

$$\limsup_{h \searrow 0} \frac{d(y_{t+h}, \text{curve}_{f(y_t)}(h))}{h} = 0.$$

Problem: how to detect if f is Lipschitz? Not trivial to compare $f(x) \in \mathrm{T}_x\,\Omega$ and $f(y) \in \mathrm{T}_y\,\Omega$.





Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023).

Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023). The idea is to take the desired property as a definition:

- assume from the beginning that a “velocity” is the evaluation at x of a globally defined object, termed a “transition”.

Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023). The idea is to take the desired property as a definition:

- assume from the beginning that a "velocity" is the evaluation at x of a globally defined object, termed a "transition". This is a semigroup $\theta : \mathbb{R}^+ \times \Omega \to \Omega$ with Lipschitz assumptions, most important one being

$$\limsup_{h \searrow 0} \frac{d\left(\theta(h,x), \theta(h,y)\right) - d(x,y)}{h} \leqslant \alpha d(x,y).$$

Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023). The idea is to take the desired property as a definition:

- assume from the beginning that a "velocity" is the evaluation at x of a globally defined object, termed a "transition". This is a semigroup $\theta : \mathbb{R}^+ \times \Omega \to \Omega$ with Lipschitz assumptions, most important one being

$$\limsup_{h \searrow 0} \frac{d\left(\theta(h,x), \theta(h,y)\right) - d(x,y)}{h} \leqslant \alpha d(x,y).$$

  Then the "velocity v" is associated to the curve $h \mapsto \theta(h,x)$.

Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023). The idea is to take the desired property as a definition:

- assume from the beginning that a "velocity" is the evaluation at x of a globally defined object, termed a "transition". This is a semigroup $\theta : \mathbb{R}^+ \times \Omega \to \Omega$ with Lipschitz assumptions, most important one being
$$\limsup_{h \searrow 0} \frac{d\left(\theta(h,x), \theta(h,y)\right) - d(x,y)}{h} \leqslant \alpha d(x,y).$$
Then the "velocity v" is associated to the curve $h \mapsto \theta(h,x)$.
- **Transitions are a metric space** with $\mathbb{D}(\theta,\theta') := \sup_{x\in\Omega} \limsup_{h\searrow 0} d(\theta(h,x), \theta'(h,x))/h$.

Mutations

Precisely to solve such problems: theory of mutations (Aubin 1999, Lorenz 2010, Frankowska-Lorenz 2023). The idea is to take the desired property as a definition:

- assume from the beginning that a "velocity" is the evaluation at x of a globally defined object, termed a "transition". This is a semigroup $\theta : \mathbb{R}^+ \times \Omega \to \Omega$ with Lipschitz assumptions, most important one being
$$\limsup_{h \searrow 0} \frac{d\left(\theta(h,x), \theta(h,y)\right) - d(x,y)}{h} \leqslant \alpha d(x,y).$$
Then the "velocity v" is associated to the curve $h \mapsto \theta(h,x)$.
- **Transitions are a metric space** with $\mathbb{D}(\theta, \theta') := \sup_{x \in \Omega} \limsup_{h \searrow 0} d(\theta(h,x), \theta'(h,x))/h$.
- Extensive theory of existence, well-posedness, estimates par to Cauchy-Lipschitz, Filippov theorems, invariance and viability, application to control of sets...

Mutations in CAT(0) spaces

Still, one has to provide the transitions.

Mutations in CAT(0) spaces

Still, one has to provide the transitions. "Change of variable":

start from x and follow the direction v

Mutations in CAT(0) spaces

Still, one has to provide the transitions. "Change of variable":

~~start from x and follow the direction v~~

start from x towards the point $\exp_x(v) \simeq x + v$.

Mutations in CAT(0) spaces

Still, one has to provide the transitions. "Change of variable":

~~start from x and follow the direction v~~

start from x towards the point $\exp_x(v) \simeq x + v$.

Definition – Transitions in CAT(0) spaces For any $\alpha \geqslant 0$ and $x_0 \in \Omega$, define a transition θ_{α,x_0} as the gradient flow of $\alpha d(\cdot, x_0)$.

Mutations in CAT(0) spaces

Still, one has to provide the transitions. "Change of variable":

~~start from x and follow the direction v~~

start from x towards the point $\exp_x(v) \simeq x + v$.

Definition – Transitions in CAT(0) spaces For any $\alpha \geqslant 0$ and $x_0 \in \Omega$, define a transition θ_{α,x_0} as the gradient flow of $\alpha d(\cdot, x_0)$.

Theorem These flows indeed provide transitions in the sense of mutations.

Mutations in CAT(0) spaces

Still, one has to provide the transitions. "Change of variable":

~~start from x and follow the direction v~~

start from x towards the point $\exp_x(v) \simeq x + v$.

Definition – Transitions in CAT(0) spaces For any $\alpha \geqslant 0$ and $x_0 \in \Omega$, define a transition θ_{α,x_0} as the gradient flow of $\alpha d(\cdot, x_0)$.

Theorem These flows indeed provide transitions in the sense of mutations.

Possible (and fruitful) to consider other families than $(\alpha, x_0) \mapsto \alpha d(\cdot, x_0)$.

Well-posedness of controlled systems

For the sequel, U is a compact set of controls.

Definition – Controlled ODE A controlled "vector field" is an application $f : \Omega \times U \to \Theta$, where Θ is the set of transitions defined before.

Well-posedness of controlled systems

For the sequel, U is a compact set of controls.

Definition – Controlled ODE A controlled "vector field" is an application $f : \Omega \times U \to \Theta$, where Θ is the set of transitions defined before. If f is Lipschitz in both variables,

Well-posedness of controlled systems

For the sequel, U is a compact set of controls.

Definition – Controlled ODE A controlled "vector field" is an application $f : \Omega \times U \to \Theta$, where Θ is the set of transitions defined before. If f is Lipschitz in both variables, then for any $u \in L^0([0,T];U)$ and $x \in \Omega$, there exists a unique absolutely continuous curve $y : [0,T] \to \Omega$ such that

$$\limsup_{h \searrow 0} \frac{d\,(y_{t+h}, f(y_t, u(t))(h, y_t))}{h} = 0 \qquad \text{for a.e. } t \in [0,T].$$

Well-posedness of controlled systems

For the sequel, U is a compact set of controls.

Definition – Controlled ODE A controlled "vector field" is an application $f : \Omega \times U \to \Theta$, where Θ is the set of transitions defined before. If f is Lipschitz in both variables, then for any $u \in L^0([0,T];U)$ and $x \in \Omega$, there exists a unique absolutely continuous curve $y : [0,T] \to \Omega$ such that

$$\limsup_{h \searrow 0} \frac{d\left(y_{t+h}, f(y_t, u(t))(h, y_t)\right)}{h} = 0 \qquad \text{for a.e. } t \in [0,T].$$

Here $\theta := f(y_t, u(t))$ is a transition, hence a semigroup, and $\theta(h, y_t)$ is the associated curve starting from y_t and evolving until time h.

Control problem

Now that we have controlled systems, we can consider *control problems*, for instance

$$\begin{cases} \text{Minimize} \quad \mathfrak{J}(y_T^{x,u}) \text{ over controls } u \in L^0([0,T];U) \\ \text{where } (y_t^{x,u})_t \text{ solves the previous system with control } u(\cdot). \end{cases} \tag{CP}$$

Control problem

Now that we have controlled systems, we can consider *control problems*, for instance

$$\begin{cases} \text{Minimize} \quad \mathfrak{J}(y_T^{x,u}) \text{ over controls } u \in L^0([0,T];U) \\ \text{where } (y_t^{x,u})_t \text{ solves the previous system with control } u(\cdot). \end{cases} \tag{CP}$$

Let V be the value function of (CP):

$$V(t,x) := \inf_{u \in L^0([t,T];U)} \mathfrak{J}\left(y_T^{t,x,u}\right).$$

Control problem

Now that we have controlled systems, we can consider *control problems*, for instance

$$\begin{cases} \text{Minimize} \quad \mathfrak{J}(y_T^{x,u}) \text{ over controls } u \in L^0([0,T];U) \\ \text{where } (y_t^{x,u})_t \text{ solves the previous system with control } u(\cdot). \end{cases} \tag{CP}$$

Let V be the value function of (CP):

$$V(t,x) := \inf_{u \in L^0([t,T];U)} \mathfrak{J}\left(y_T^{t,x,u}\right).$$

Using the results of mutational analysis, the Dynamical Programming Principle and the local Lipschitz character of V follow exactly as in the classical case.

Control problem

Now that we have controlled systems, we can consider *control problems*, for instance

$$\begin{cases} \text{Minimize} \quad \mathfrak{J}(y_T^{x,u}) \text{ over controls } u \in L^0([0,T];U) \\ \text{where } (y_t^{x,u})_t \text{ solves the previous system with control } u(\cdot). \end{cases} \tag{CP}$$

Let V be the value function of (CP):

$$V(t,x) := \inf_{u \in L^0([t,T];U)} \mathfrak{J}\left(y_T^{t,x,u}\right).$$

Using the results of mutational analysis, the Dynamical Programming Principle and the local Lipschitz character of V follow exactly as in the classical case. What about HJ?

Hamilton-Jacobi-Bellman equation

If φ is directionally differentiable anywhere*, and $\theta \in \Theta$, let $L_\theta \varphi$ be the directional derivative

$$L_\theta \varphi(x) := \lim_{h \searrow 0} \frac{\varphi(\theta(h,x)) - \varphi(x)}{h}.$$

Hamilton-Jacobi-Bellman equation

If φ is directionally differentiable anywhere*, and $\theta \in \Theta$, let $L_\theta \varphi$ be the directional derivative

$$L_\theta \varphi(x) := \lim_{h \searrow 0} \frac{\varphi(\theta(h,x)) - \varphi(x)}{h}.$$

To a dynamic $f : \Omega \times U \to \Theta$, associate a Hamiltonian

$$H(x, \varphi) := \sup_{u \in U} -L_{f(x,u)} \varphi(x).$$

Hamilton-Jacobi-Bellman equation

If φ is directionally differentiable anywhere*, and $\theta \in \Theta$, let $L_\theta \varphi$ be the directional derivative

$$L_\theta \varphi(x) := \lim_{h \searrow 0} \frac{\varphi(\theta(h,x)) - \varphi(x)}{h}.$$

To a dynamic $f : \Omega \times U \to \Theta$, associate a Hamiltonian

$$H(x,\varphi) := \sup_{u \in U} -L_{f(x,u)} \varphi(x).$$

Theorem – [AZ25] The value function is the unique viscosity solution of the HJ equation

$$-\partial_t u(t,x) + H(x, D_x u(t,x)) = 0 \quad \text{in } (0,T) \times \Omega, \qquad u(T,\cdot) = \mathfrak{J}.$$

Table of Contents

Classical setting

Not all control problems are well-posed (in the variable u); consider

$$\begin{cases} \text{Minimize } |y_T^{x,u}|^2 \text{ over all } u \in L^0([0,T];\{-1,1\}), \\ \text{where } \frac{d}{dt} y_t^{x,u} = u(t) \text{ and } y_0^{x,u} = 0. \end{cases}$$

Classical setting

Not all control problems are well-posed (in the variable u); consider

$$\begin{cases} \text{Minimize } |y_T^{x,u}|^2 \text{ over all } u \in L^0([0,T];\{-1,1\}), \\ \text{where } \frac{d}{dt} y_t^{x,u} = u(t) \text{ and } y_0^{x,u} = 0. \end{cases}$$

A minimizing sequence would try to "not move", by alternating rapidly between $+1$ and -1.

Classical setting

Not all control problems are well-posed (in the variable u); consider

$$\begin{cases} \text{Minimize } |y_T^{x,u}|^2 \text{ over all } u \in L^0([0,T];\{-1,1\}), \\ \text{where } \frac{d}{dt} y_t^{x,u} = u(t) \text{ and } y_0^{x,u} = 0. \end{cases}$$

A minimizing sequence would try to "not move", by alternating rapidly between $+1$ and -1.

☛ Need to convexify the set of admissible dynamics to obtain existence.

Classical setting

Not all control problems are well-posed (in the variable u); consider

$$\begin{cases} \text{Minimize } |y_T^{x,u}|^2 \text{ over all } u \in L^0([0,T];\{-1,1\}), \\ \text{where } \frac{d}{dt} y_t^{x,u} = u(t) \text{ and } y_0^{x,u} = 0. \end{cases}$$

A minimizing sequence would try to “not move”, by alternating rapidly between $+1$ and -1.

☛ Need to convexify the set of admissible dynamics to obtain existence.
Here, one would replace $\{-1,1\}$ by $[-1,1]$, and the control $u(\cdot) \equiv 0$ is optimal.

CAT(0) spaces

In CAT(0) spaces, what is the appropriate notion of "convexification"?

(a) Tangent cones $\mathrm{T}_x \Omega$ are themselves CAT(0) spaces, in which barycenters are defined (Sturm 2003).

CAT(0) spaces

In CAT(0) spaces, what is the appropriate notion of "convexification"?

(a) Tangent cones $\mathrm{T}_x\,\Omega$ are themselves CAT(0) spaces, in which barycenters are defined (Sturm 2003).

(b) Taking "convex combinations" of semigroups: for instance by a Trotter scheme, following alternatively each one with increasing frequency?

CAT(0) spaces

In CAT(0) spaces, what is the appropriate notion of "convexification"?

(a) Tangent cones $\mathrm{T}_x\,\Omega$ are themselves CAT(0) spaces, in which barycenters are defined (Sturm 2003).

(b) Taking "convex combinations" of semigroups: for instance by a Trotter scheme, following alternatively each one with increasing frequency?

(c) Since we have gradient flows, follow the gradient flow of the convex combination?

CAT(0) spaces

In CAT(0) spaces, what is the appropriate notion of "convexification"?

(a) Tangent cones $\mathrm{T}_x\,\Omega$ are themselves CAT(0) spaces, in which barycenters are defined (Sturm 2003).

(b) Taking "convex combinations" of semigroups: for instance by a Trotter scheme, following alternatively each one with increasing frequency?

(c) Since we have gradient flows, follow the gradient flow of the convex combination?

In fact, (c) can be used to obtain existence, and implies both (a) and (b).

Proper definition

Definition Let $f : \Omega \times U \to \Theta$ be a controlled dynamic.

Proper definition

Definition Let $f : \Omega \times U \to \Theta$ be a controlled dynamic. For each $(x, u) \in \Omega \times U$, denote $E_{f(x,u)} : \Omega \to \mathbb{R}$ the function such that $f(x, u)$ is the gradient flow of $E_{f(x,u)}$.

Proper definition

Definition Let $f : \Omega \times U \to \Theta$ be a controlled dynamic. For each $(x, u) \in \Omega \times U$, denote $E_{f(x,u)} : \Omega \to \mathbb{R}$ the function such that $f(x, u)$ is the gradient flow of $E_{f(x,u)}$. To each $x \in \Omega$ and "relaxed control" $\omega \in \mathscr{P}(U)$, associate the function

$$E_\omega := \int_{u \in U} E_{f(x,u)} d\omega(u).$$

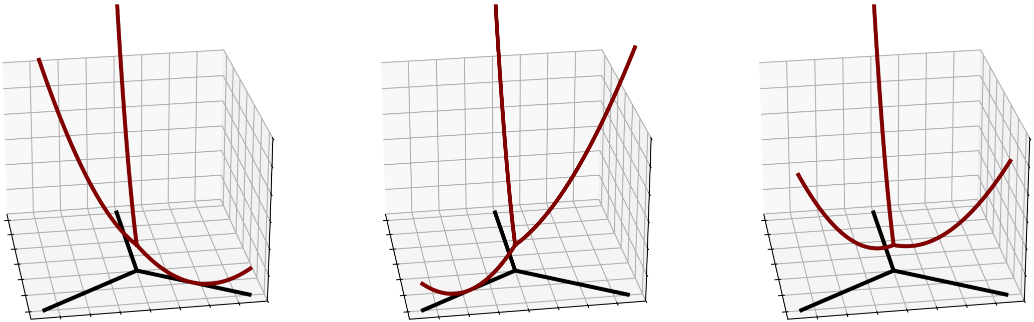
Proper definition

Definition Let $f : \Omega \times U \to \Theta$ be a controlled dynamic. For each $(x,u) \in \Omega \times U$, denote $E_{f(x,u)} : \Omega \to \mathbb{R}$ the function such that $f(x,u)$ is the gradient flow of $E_{f(x,u)}$. To each $x \in \Omega$ and "relaxed control" $\omega \in \mathscr{P}(U)$, associate the function

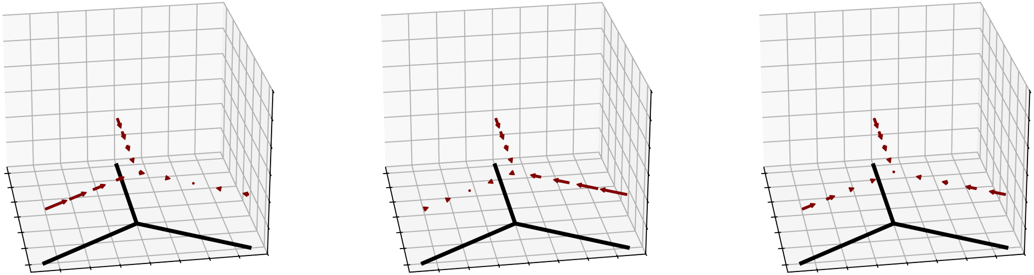
$$E_\omega := \int_{u \in U} E_{f(x,u)} d\omega(u).$$

The convexification $F : \Omega \times \mathscr{P}(U) \to \Theta_{\text{conv}}$ is defined by $F(x,\omega) := \text{Gradient Flow}(E_\omega)$.

Example: tripod network



Example: tripod network



Existence and relaxation result

Theorem The convexification still generates well-posed trajectories.

Existence and relaxation result

Theorem The convexification still generates well-posed trajectories. Moreover, if $\mathfrak{J}$ is lsc,

- the control problem posed on trajectories driven by F admits a solution in $L^0([0,T];\mathscr{P}(U))$.

Existence and relaxation result

Theorem The convexification still generates well-posed trajectories. Moreover, if $\mathfrak{J}$ is lsc,

- the control problem posed on trajectories driven by F admits a solution in $L^0([0,T];\mathscr{P}(U))$.
- the value functions of the original problem (with f) and the relaxed one (with F) coincide.

Existence and relaxation result

Theorem The convexification still generates well-posed trajectories. Moreover, if $\mathfrak{J}$ is lsc,

- the control problem posed on trajectories driven by F admits a solution in $L^0([0,T];\mathscr{P}(U))$.
- the value functions of the original problem (with f) and the relaxed one (with F) coincide.

Draws from Frankowska & Lorenz 2023.

Existence and relaxation result

Theorem The convexification still generates well-posed trajectories. Moreover, if $\mathfrak{J}$ is lsc,

- the control problem posed on trajectories driven by F admits a solution in $L^0([0,T];\mathscr{P}(U))$.
- the value functions of the original problem (with f) and the relaxed one (with F) coincide.

Draws from Frankowska & Lorenz 2023.
The link with Trotter-Kato approximations was made by Stojkovic (2012).

Existence and relaxation result

Theorem The convexification still generates well-posed trajectories. Moreover, if $\mathfrak{J}$ is lsc,

- the control problem posed on trajectories driven by F admits a solution in $L^0([0,T];\mathscr{P}(U))$.
- the value functions of the original problem (with f) and the relaxed one (with F) coincide.

Draws from Frankowska & Lorenz 2023.
The link with Trotter-Kato approximations was made by Stojkovic (2012).
When looking at a given point $x \in \Omega$, under some natural assumptions over Θ, one has that

the metric gradient of the convexification is the barycenter of the metric gradients

$$\nabla_x \left[\int_u E_u d\omega(u) \right] = \int_u \left[\nabla_x E_u \right] d\omega(u).$$

Table of Contents

Numerical schemes

Since the definition of "dynamics" comes with flows, one easily defines a (rough) semi-Lagrangian scheme for the value function, by discretizing the Bellman principle

$$V(t,x) = \inf_{u \in L^0([t,t+h];U)} V(t+h, y_{t+h}^{t,x,u}), \qquad \text{with} \qquad V(T,\cdot) = \mathfrak{J}.$$

Numerical schemes

Since the definition of "dynamics" comes with flows, one easily defines a (rough) semi-Lagrangian scheme for the value function, by discretizing the Bellman principle

$$V(t,x) = \inf_{u \in L^0([t,t+h];U)} V(t+h, y_{t+h}^{t,x,u}), \qquad \text{with} \qquad V(T,\cdot) = \mathfrak{J}.$$

Namely, compute $\hat{V}_n(\hat{x})$ for each point $\hat{x}$ of a grid, with

$$\hat{V}_N(\hat{x}) := \mathfrak{J}(\hat{x}), \qquad \text{and} \qquad \hat{V}_n(\hat{x}) := \min_{u \in U} \hat{V}_{n+1}\left(\mathrm{Proj}_{\mathsf{Grid}} f(\hat{x},u)(\Delta t, \hat{x})\right).$$

Numerical schemes

Since the definition of "dynamics" comes with flows, one easily defines a (rough) semi-Lagrangian scheme for the value function, by discretizing the Bellman principle

$$V(t,x) = \inf_{u \in L^0([t,t+h];U)} V(t+h, y^{t,x,u}_{t+h}), \qquad \text{with} \qquad V(T,\cdot) = \mathfrak{J}.$$

Namely, compute $\hat{V}_n(\hat{x})$ for each point $\hat{x}$ of a grid, with

$$\hat{V}_N(\hat{x}) := \mathfrak{J}(\hat{x}), \qquad \text{and} \qquad \hat{V}_n(\hat{x}) := \min_{u \in U} \hat{V}_{n+1}\left(\mathsf{Proj}_{\mathsf{Grid}} f(\hat{x},u)(\Delta t, \hat{x})\right).$$

- One could also consider the PDE formulation

$$-\partial_t u(t,x) + \sup_{u \in U} -L_{f(x,u)} u(t,x) = 0, \qquad u(T,\cdot) = \mathfrak{J}.$$

Numerical schemes

Since the definition of "dynamics" comes with flows, one easily defines a (rough) semi-Lagrangian scheme for the value function, by discretizing the Bellman principle

$$V(t,x) = \inf_{u \in L^0([t,t+h];U)} V(t+h, y_{t+h}^{t,x,u}), \qquad \text{with} \qquad V(T,\cdot) = \mathfrak{J}.$$

Namely, compute $\hat{V}_n(\hat{x})$ for each point $\hat{x}$ of a grid, with

$$\hat{V}_N(\hat{x}) \coloneqq \mathfrak{J}(\hat{x}), \qquad \text{and} \qquad \hat{V}_n(\hat{x}) \coloneqq \min_{u \in U} \hat{V}_{n+1}\left(\mathrm{Proj}_{\mathrm{Grid}} f(\hat{x},u)(\Delta t, \hat{x})\right).$$

- One could also consider the PDE formulation

$$-\partial_t u(t,x) + \sup_{u \in U} -L_{f(x,u)} u(t,x) = 0, \qquad u(T,\cdot) = \mathfrak{J}.$$

PDE-based schemes have been studied for gluings by Costeseque, Lebacque and Monneau 2015, Guerand and Koumaiha 2019, Morfe 2020, and Bokanowski, Jerhaoui and Zidani 2025.

Error

(A1) We assume that in the particular case of the CAT(0) space Ω, one knows how to approximate at order 2 the transitions.

Error

(A1) We assume that in the particular case of the CAT(0) space Ω, one knows how to approximate at order 2 the transitions.

In $\mathbb{R}^d$ and gluings, taking flows of distance functions, explicit computation.

Error

(A1) We assume that in the particular case of the CAT(0) space Ω, one knows how to approximate at order 2 the transitions.

In $\mathbb{R}^d$ and gluings, taking flows of distance functions, explicit computation.

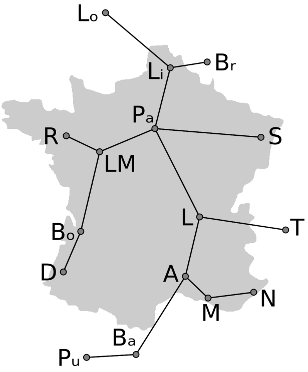
Proposition Under (A1) and Lipschitz assumption on f, there holds

$$\left|\hat{V}_n(\hat{x}) - V(t_n, \hat{x})\right| \leqslant C_{T,\mathrm{Lip}(V)} \left(\frac{\Delta x}{\Delta t} + \Delta t\right).$$

Hence the scheme is convergent with order 1/2 under the inverse CFL condition $\Delta t = \sqrt{\Delta x}$.

Example: the SNCF network

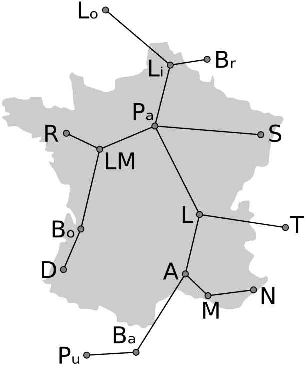
Good example of a CAT(0) 1D network (no loop).



Example: the SNCF network

Good example of a CAT(0) 1D network (no loop).

Dynamics are allowing the user to get closer to Londres, Madrid (Puerta de Atocha), Nice or Dax, with speed limit restrictions near Marseille, and an increasingly efficient network in Aquitaine.

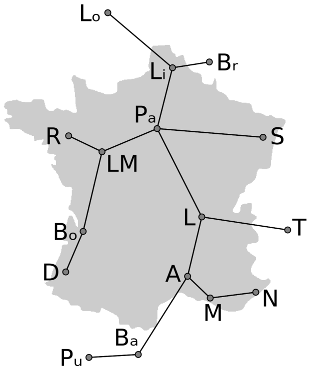


Example: the SNCF network

Good example of a CAT(0) 1D network (no loop).

Dynamics are allowing the user to get closer to Londres, Madrid (Puerta de Atocha), Nice or Dax, with speed limit restrictions near Marseille, and an increasingly efficient network in Aquitaine.

The cost penalizes the distance to Bordeaux, Marseille or Barcelona (aestival problem). The exact value function can be computed.

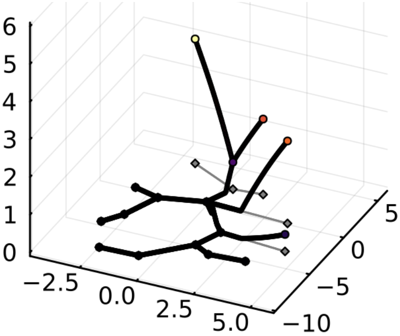


Example: the SNCF network

Good example of a CAT(0) 1D network (no loop).

Dynamics are allowing the user to get closer to Londres, Madrid (Puerta de Atocha), Nice or Dax, with speed limit restrictions near Marseille, and an increasingly efficient network in Aquitaine.

The cost penalizes the distance to Bordeaux, Marseille or Barcelona (aestival problem). The exact value function can be computed.

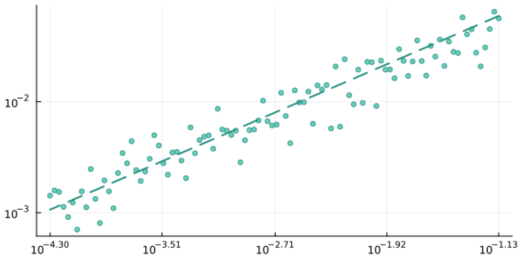


Example: the SNCF network

Good example of a CAT(0) 1D network (no loop).

Dynamics are allowing the user to get closer to Londres, Madrid (Puerta de Atocha), Nice or Dax, with speed limit restrictions near Marseille, and an increasingly efficient network in Aquitaine.

The cost penalizes the distance to Bordeaux, Marseille or Barcelona (aestival problem). The exact value function can be computed.



Relative error on the value function w.r.t. Δx. The slope is 0.55.

Example: a robust problem (1/2)

Consider a problem in dimension $d = 2$, with G sym. def. pos:

$$\begin{cases} \text{Minimize } \max_{x \in \overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u} \rangle \text{ over controls } u \in L^1(0,T;U), \\ \text{where } y_0^{x,u} = x \text{ and } \dot{y}_t^{x,u} = A(u(t)) y_t^{x,u}. \end{cases}$$

Example: a robust problem (1/2)

Consider a problem in dimension $d = 2$, with G sym. def. pos:

$$\begin{cases} \text{Minimize } \max_{x \in \overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u} \rangle \text{ over controls } u \in L^1(0,T;U), \\ \text{where } y_0^{x,u} = x \text{ and } \dot{y}_t^{x,u} = A(u(t)) y_t^{x,u}. \end{cases}$$

To any control, associate the resolvent $R^u : [0,T] \to \mathbb{M}_{2,2}$ that satisfies $\dot{R}^u = A(u) R^u$.

Example: a robust problem (1/2)

Consider a problem in dimension $d = 2$, with G sym. def. pos:

$$\begin{cases} \text{Minimize } \max_{x\in\overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u}\rangle \text{ over controls } u \in L^1(0,T;U), \\ \text{where } y_0^{x,u} = x \text{ and } \dot{y}_t^{x,u} = A(u(t)) y_t^{x,u}. \end{cases}$$

To any control, associate the resolvent $R^u : [0,T] \to \mathbb{M}_{2,2}$ that satisfies $\dot{R}^u = A(u)R^u$. Then there holds $y_t^{x,u} = R^u(t)x$, so that

$$\max_{x\in\overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u}\rangle = \max_{x\in\overline{\mathscr{B}}(0,1)} \langle (R^u)^*(t) G R^u(t) x, x\rangle = \lambda_{\max}\left((R^u)^*(t) G R^u(t)\right).$$

Example: a robust problem (1/2)

Consider a problem in dimension $d = 2$, with G sym. def. pos:

$$\begin{cases} \text{Minimize } \max_{x\in\overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u}\rangle \text{ over controls } u \in L^1(0,T;U), \\ \text{where } y_0^{x,u} = x \text{ and } \dot{y}_t^{x,u} = A(u(t))y_t^{x,u}. \end{cases}$$

To any control, associate the resolvent $R^u : [0,T] \to \mathbb{M}_{2,2}$ that satisfies $\dot{R}^u = A(u)R^u$. Then there holds $y_t^{x,u} = R^u(t)x$, so that

$$\max_{x\in\overline{\mathscr{B}}(0,1)} \langle G y_T^{x,u}, y_T^{x,u}\rangle = \max_{x\in\overline{\mathscr{B}}(0,1)} \langle (R^u)^*(t) G R^u(t)x, x\rangle = \lambda_{\max}\left((R^u)^*(t) G R^u(t)\right).$$

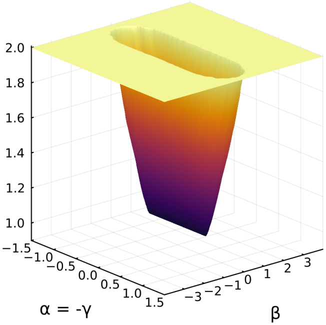
☛ Classical control problem with cost $\mathfrak{J} = \lambda_{\max}$, on trajectories of sdp matrices satisfying the ODE

$$G^u(0) = G, \qquad \dot{G}^u(t) = A^*(u(t))G^u(t) + G^u(t)A(u(t)).$$

Example: a robust problem (2/2)

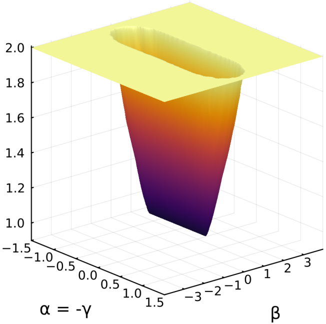
SDP matrices can be metrized as a CAT(0) space (Bhatia 2007), so we can apply the previous theory.

Example: a robust problem (2/2)

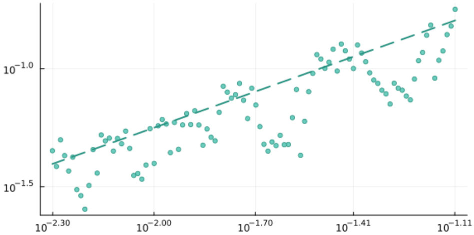


SDP matrices can be metrized as a CAT(0) space (Bhatia 2007), so we can apply the previous theory.

Example: a robust problem (2/2)

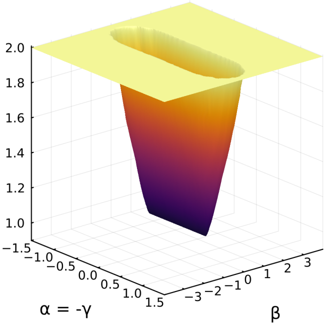


SDP matrices can be metrized as a CAT(0) space (Bhatia 2007), so we can apply the previous theory.

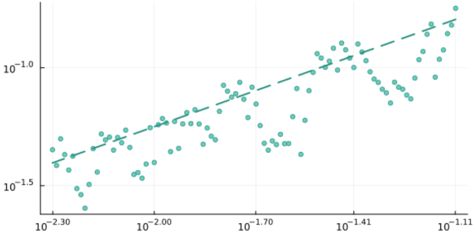


Relative error on the value function w.r.t. Δx. The slope is 0.627.

Example: a robust problem (2/2)



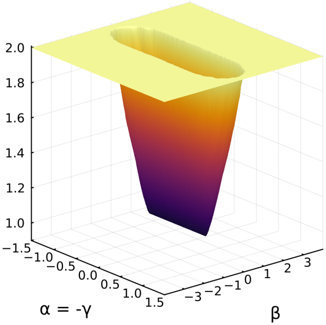
SDP matrices can be metrized as a CAT(0) space (Bhatia 2007), so we can apply the previous theory.



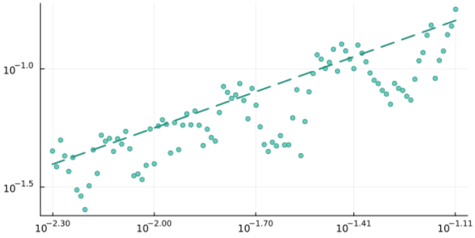
Relative error on the value function w.r.t. Δx. The slope is 0.627.

- Quite heavy numerically: computation of geodesics, meshes, semi-Lagrangian itself...

Example: a robust problem (2/2)



SDP matrices can be metrized as a CAT(0) space (Bhatia 2007), so we can apply the previous theory.



Relative error on the value function w.r.t. Δx. The slope is 0.627.

- Quite heavy numerically: computation of geodesics, meshes, semi-Lagrangian itself...
- The projection on the grid should be replaced by a higher-order method (classically, interpolation): not trivial in general CAT(0) spaces.

Open directions

Interesting points include:

- Original aim of this work: populations over networks.

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?
- Letting the underlying space change.

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?
- Letting the underlying space change. Discontinuity of Lip around networks?

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?
- Letting the underlying space change. Discontinuity of Lip around networks? Systematic regularization procedure?

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?
- Letting the underlying space change. Discontinuity of Lip around networks? Systematic regularization procedure? Justification of Imbert and Monneau's transition function?

Open directions

Interesting points include:

- Original aim of this work: populations over networks. Significantly harder to work in spaces of measures over CAT(0) spaces, even at the HJ level: how to get good test functions?
- Letting the underlying space change. Discontinuity of Lip around networks? Systematic regularization procedure? Justification of Imbert and Monneau's transition function?

Thank you for you attention.

Talk based on the preprint "A Cauchy-Lipschitz setting for control problems in complete CAT(0) spaces", with H. Zidani.
Details of numerical examples are given in my PhD manuscript (available on HAL theses),
check also Othmane's for details of theoretical results on HJ equations.

Julia code available on `github.com/averil-aussedat/FLagHada.jl`.