(Max,+)

Understanding Hamilton-Jacobi as Maslov processes

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Dedicated to Виктор Маслов, who unfortunately left us on August 3rd, 2023.

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Kαfεmιnαrιo - Universidad Santa María

Table of Contents

The (max,+) idempotent calculus

We consider $\mathbb{R} \cup \{-\infty\}$ endowed with the following operations:

$$a \oplus b := \max(a, b), \qquad a \otimes b := a + b.$$

Both operations are commutative and associative, and

$$a \otimes (b \oplus c) = a + \max(b, c) = \max(a + b, a + c) = (a \otimes b) \oplus (a \otimes c).$$

Define $\mathbb{0} := -\infty$ and $\mathbb{1} := 0$. Then

$$\mathbb{0} \oplus a = \max(-\infty, a) = a, \qquad \mathbb{0} \otimes a = -\infty + a = \mathbb{0}, \qquad \mathbb{1} \otimes a = a + 0 = a.$$

Then $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is a semiring (ring without additive inverse). The name idempotent comes from the fact that $a \oplus a = a$.

Examples

Define the (max,+) division $\oslash$ by

$$a \oslash b := a - b.$$

As in the classical algebra, one can't divide by $\mathbb{0}$. With this notation, the classical positive and negative parts of a number become

$$a_+ = \max(0, a) = \mathbb{1} \oplus a, \qquad a_- = \max(0, -a) = \mathbb{1} \oplus \left(\mathbb{1} \oslash a\right).$$

One may go further and define the (max,+) equivalents of the sum and integrals

$$\sum\nolimits^{\oplus} a_i := \max_{i \in [\![1,n]\!]} a_i, \qquad \text{and} \qquad \int_{\lambda \in \Lambda}^{\oplus} a_\lambda := \sup_{\lambda \in \Lambda} a_\lambda.$$

Table of Contents

Definition

Def 1 – Maslov measure [KM97, in text, p.36] Let X be an Hausdorff and locally compact space, and $\mathcal{X} \subset \mathcal{P}(X)$ a $\sigma-$algebra. A Maslov measure is a map $\mu : \mathcal{X} \to \overline{\mathbb{R}}$ satisfying

$$\mu\left(\bigcup_{a\in A} B_a\right) = \int_{a\in A}^{\oplus} \mu(B_a) = \sup_{a\in A} \mu(B_a) \tag{1}$$

for any family of sets $(B_a)_{a\in A} \subset \mathcal{X}$.

From (1), one deduces that $\mu(\emptyset) = \mathbb{0}$. A Maslov measure is *bounded* if $\mu(X) \in \mathbb{R} \cup \{-\infty\}$. By definition, the measure μ is monotone, in the sense that $\mu(B) \leqslant \mu(X)$ for all B: indeed,

$$\mu(X) = \mu(B \cup (X \setminus B)) = \mu(B) \oplus \mu(X \setminus B) = \max\left(\mu(B), \mu(X \setminus B)\right).$$

Representation (1/2)

Consider $f : X \to \mathbb{R} \cup \{-\infty\}$ upper bounded, and

$$\mu(B) := \int_{x \in B}^{\oplus} f(x) = \sup_{x \in B} f(x).$$

Then for any family $(B_a)_{a \in A} \subset \mathcal{X}$,

$$\mu\left(\bigcup_{a \in A} B_a\right) = \sup_{x \in \bigcup_{a \in A} B_a} f(x) = \sup_{a \in A} \sup_{x \in B_a} f(x) = \int_{a \in A}^{\oplus} \int_{x \in B_a}^{\oplus} f(x) = \int_{a \in A}^{\oplus} \mu(B).$$

Moreover,

$$\mu(X) = \sup_{x \in X} f(x) \leqslant [\![f]\!],$$

so that μ is a bounded Maslov measure.

Representation (2/2)

Conversely, consider μ a bounded Maslov measure, and define

$$f : X \to \mathbb{R} \cup \{-\infty\}, \qquad f(x) := \mu\left(\{x\}\right).$$

Then for any set $B \in \mathcal{X}$,

$$\mu\left(B\right) = \mu\left(\bigcup_{x \in B} \{x\}\right) = \int_{x \in B}^{\oplus} \mu(\{x\}) = \int_{x \in B}^{\oplus} f(x).$$

Moreover $f(x) = \mu\left(\{x\}\right) \leqslant \mu(X)$, and f is upper bounded.

> All Maslov measures admit a *density* $f : X \to \mathbb{R} \cup \{-\infty\}$ such that $\mu(B) = \int_{x \in B}^{\oplus} f(x)$.

Definition

Def 2 – Maslov random variable Let $(X, \mathcal{X}, \mu)$ a Maslov measured space, and $(E, \mathcal{E})$ a measurable space. A random variable is a $(\mathcal{X}, \mathcal{E})-$measurable map $V : X \to E$.

Not the only definition ([DMD99, Definition 3] asks for a continuity condition), but simple. For any random variable V, define the *law* $\mu_V := V\#\mu$ as

$$\mu_V : \mathcal{E} \to \mathbb{R} \cup \{-\infty\}, \qquad \mu_V(B) := (V\#\mu)(B) = \mu\left(V^{-1}(B)\right) = \mu\left(\{x \in X \mid V(x) \in B\}\right).$$

Then

$$\mu_V(B) = \sup_{x \in V^{-1}(B)} f(x) = \sup_{a \in B} \inf_{x \in X, V(x) = a} f(x) = \int_{a \in B}^{\oplus} g(a),$$

where $g : E \to \mathbb{R} \cup \{-\infty\}$ is given by $g(a) = \sup_{x \in X, V(x) = a} f(x) = \mu\left(V^{-1}(\{a\})\right)$. As μ_V is trivially upper bounded, μ_V is a bounded Maslov measure on $(E, \mathcal{E})$.

Example

Consider $X = \mathbb{R}$, the Maslov measure μ of density $f(x) = -|x|$ and the random variable

$$V : \mathbb{R} \to \mathbb{R}, \qquad V(x) = \sin(x).$$

Then for all $B \subset X$,

$$\mu_V(B) = \mu(\{x \in \mathbb{R} \mid \sin(x) \in B\}) = \sup_{x \in \mathbb{R}, \sin(x) \in B} -|x|,$$

and the density of μ_V is

$$f_V(y) = \mu_V(\{y\}) = \sup_{x \in \sin^{-1}(\{y\})} -|x| = \begin{cases} -|\arcsin(y)| & \text{if } y \in [-1, 1], \\ \mathbb{0} & \text{otherwise.} \end{cases}$$

Table of Contents

Stochastic process

Let $(X, \mathcal{X}, \mu)$ be a Maslov measured space. Let $(\mathcal{F}_t)_t$ be a filtration over a metric space (E, d).

Def 3 – Maslov stochastic process (inspired from [DMD99, Definition 7]) A Maslov stochastic process over $[0, T]$ with values in a metric space (E, d) is a family $P = (P_t)_{t\in[0,T]}$ of Maslov random variables $P_t : X \to E$ such that each P_t is $(\mathcal{X}, \mathcal{F}_t)$ −measurable.

The interpretation of P is that

- $t \mapsto P_t(x)$ is a curve in E,
- $x \mapsto P_t(x)$ is the state of P at time t.

For instance, X could be $\left\{(x_0, v) \mid x_0 \in \mathbb{R}^d,\ v \in \mathsf{T}_{x_0}\mathbb{R}^d\right\}$, and $P_t(x) = P_t(x_0, v)$ be the point $x_0 + tv$, i.e. the evaluation at time t of the trajectory issued from x_0 following the control v.

Maslov chains

Consider again $(X, \mathcal{X}, \mu)$ a Maslov measured space. Recall that

$$\mu(A \mid B) = \mu(A \cap B) \oslash \mu(B) = \mu(A \cap B) - \mu(B)$$

by the Maslov-Bayes formula.

Def 4 – Maslov chain (Freely inspired from [DMD99, Definition 8]) Let $P = (P_t)_t$ be a stochastic process. P is a Maslov chain if for any $0 \leqslant t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$ and $A_i \in \mathcal{F}_{t_i}$ for $i \in [\![0, n]\!]$, there holds

$$\mu\left(P_{t_n} \in A_n \mid P_{t_{n-1}} \in A_{n-1} \cap \cdots \cap P_{t_0} \in A_0\right) = \mu\left(P_{t_n} \in A_n \mid P_{t_{n-1}} \in A_{n-1}\right).$$

Backward equations (1/2)

Let $J : E \to \mathbb{R} \cup \{-\infty\}$ be an arbitrary map, and denote

$$u(t,x) = \int_{y \in E}^{\oplus} J(y) \otimes \mu \left\{ P_T = y \mid P_t = x \right\}.$$

Under the previous notations, there holds for all $0 \leqslant t \leqslant \tau \leqslant T$ that

$$\begin{aligned}
u(t,x) &= \int_{y \in E}^{\oplus} J(y) \otimes \mu \left\{ P_T = y \cap P_t = x \right\} \mathbin{\overset{\circ}{/}} \mu\{P_t = x\} \\
&= \int_{z \in E}^{\oplus} \int_{y \in E}^{\oplus} J(y) \otimes \mu \left\{ P_T = y \cap P_\tau = z \cap P_t = x \right\} \mathbin{\overset{\circ}{/}} \mu\{P_t = x\} \\
&= \int_{z \in E}^{\oplus} \int_{y \in E}^{\oplus} J(y) \otimes \mu \left\{ P_T = y \mid P_\tau = z \right\} \otimes \mu \left\{ P_\tau = z \cap P_t = x \right\} \mathbin{\overset{\circ}{/}} \mu\{P_t = x\} \\
&= \int_{z \in E}^{\oplus} u(\tau, z) \otimes \mu \left\{ P_\tau = z \mid P_t = x \right\}.
\end{aligned}$$

Backward equations (2/2)

More explicitely, there holds for all $h \in [0, T-t]$ that

$$u(t,x) = \int_{z\in E}^{\oplus} u(t+h,z) \otimes \mu\left\{P_{t+h} = z \mid P_t = x\right\} = \sup_{z\in E}\left[u(t+h,z) + \mathcal{L}_{t,t+h}(x,z)\right], \quad (2)$$

where

$$\mathcal{L}_{t,t+h}(x,z) := \mu\left\{P_{t+h} = z \mid P_t = x\right\}.$$

Notice moreover that

$$u(T,x) = \int_{y\in E}^{\oplus} J(y) \otimes \mu\left\{P_T = y \mid P_T = x\right\} = J(x).$$

The link with Hamilton-Jacobi-Bellman equations

In the context of Hamilton-Jacobi-Bellman equations, (2) is called the *Dynamic Programming Principle*. It is the equation satisfied by the *value function* u of a *control problem* written as

$$\text{Find } \alpha^* \in L^0([0,T];A) \text{ maximizing } \alpha \mapsto \int_{r=0}^{T} \ell(r, \gamma_r^{0,x,\alpha}, \alpha(r))dr + J(\gamma_T^{0,x,\alpha}).$$

The map J is the *terminal cost* of the control problem, the curves $\left(\gamma_r^{0,x,\alpha}\right)_{s\in[0,T]}$ are the *trajectories* (usually solutions of $\frac{d}{dt}\gamma_t = f(t,\gamma_t,\alpha(t))$) and the map $\mathcal{L}$ is given by

$$\mathcal{L}_{t,t+h}(x,z) := \sup_{\alpha\in L^0([t,t+h];A),\gamma_{t+h}^{t,x,\alpha}=z} \int_{r=t}^{t+h} \ell(r,\gamma_r^{t,x,\alpha},\alpha(r))dr.$$

The quantity $\mathcal{L}_{t,t+h}(x,z)$ is interpreted as the maximal gain achievable from (t,x) to $(t+h,z)$.

More HJB

Under regularity assumptions over the dynamical system and J, it is shown that u is the *viscosity solution* of the HJB equation

$$-\partial_t u(t,x) + \inf_{\alpha\in A} - \langle \nabla u(t,x), f(t,x,\alpha)\rangle = 0, \qquad u(T,x) = J(x). \tag{3}$$

In the $(\max,+)-$algebra, the equation is *linear*: there holds that

$$u_0, u_1 \text{ solutions of (3) for } J_0, J_1 \implies a\otimes u_0 \oplus u_1 \text{ solution of (3) for } a\otimes J_0\oplus J_1.$$

Moreover the viscosity solution is known to be the *maximal subsolution*, i.e. the largest of the maps that satisfy

$$v(t,x) \leqslant \int_{y\in E}^{\oplus} \mathcal{L}_{t,t+h}(x,y)\otimes v(t+h,y) \qquad \forall h\in[0,T-t],\ x\in E.$$

Table of Contents

High time to conclude

In this talk, we have seen

- the definition of the $(\max, +)$ "algebra", and the associated Maslov measures,
- the Maslov-Chapman-Kolmogorov equations for Maslov stochastic processes,
- that Hamilton-Jacobi equations are satisfied by the conditional expectations of the Maslov stochastic processes.

The theory of idempotent calculus is quite widely developed, with

- its own set of linear equations: Hamilton-Jacobi-Bellman.
- its own heat equation: the Eikonal equation.
- its own weak formulation by "duality" with the "linear" applications.
- its own numerical methods (tropical finite elements!)

But that exceeds by far the content of this talk...

Thank you!

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