Local characterization of tangent plans that are martingale plans

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First notations

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$$\mathscr{P}_2(\mathrm{T}\,\Omega)_\mu \coloneqq \{\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega) \mid \pi_{x\#}\xi = \mu\}\,.$$

Any $\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ can be disintegrated as $\xi = \xi_x \otimes \mu$, called "measure field" instead of "vector field".

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W_μ comes with its "scalar product"

$$\langle \xi,\zeta\rangle_\mu := \frac{1}{2}\left[W_\mu^2(\xi,0_\mu) + W_\mu^2(\zeta,0_\mu) - W_\mu^2(\xi,\zeta)\right] = \int_{x\in\Omega} \sup_{\alpha_x\in\Gamma(\xi_x,\zeta_x)} \int_{(v,w)} \langle v,w\rangle\, d\alpha_x(v,w)d\mu(x).$$

Some literature on this pseudo-Hilbertian structure

- Introduced in [AGS05][1] and [Gig08][2] along with a scalar multiplication and a set-valued sum.

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- Recent work [LTD24][10] providing KKT conditions.

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[10] N. Lanzetti, A. Terpin, and F. Dörfler, *Variational Analysis in the Wasserstein Space* (2024).

Orthogonality

If $\xi = (id, f)_\# \mu$ for some $f \in L^2_\mu(\Omega; \mathbb{R}^d)$, then $\xi_x = \delta_{f(x)}$, so that

$$\langle \xi, \zeta \rangle_\mu = \int_{x \in \Omega} \int_w \langle f(x), w \rangle \, d\zeta_x(w) d\mu(x) = \langle f, \text{Bary}\,(\zeta) \rangle_{L^2_\mu} .$$

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Observation $\mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ splits orthogonally into

- the set of ξ that are induced by a map,
- the set of ξ with barycenter 0, noted $\mathscr{P}_2(\mathrm{T}\,\Omega)^0_\mu$.

$\mathrm{Bary}\,(\xi) = 0$ iff $(\pi_x, \pi_x + \pi_v)_{\#}\xi$ is a "martingale plan".

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On $\mathscr{P}_2(\mathrm{T}\,\Omega)^0_\mu$, very strong property:

$$\langle \xi, \zeta \rangle_\mu = 0$$

if and only if $\langle \xi_x, \zeta_x \rangle_{\delta_x} = 0$ for $\mu-$a.e. x.

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For centred fields, orthogonality is a local phenomenon.

Closed convex cone of centred fields

Theorem [Aus25][1] Let $A \subset \mathscr{P}_2(\mathrm{T}\,\Omega)^0_\mu$ be a $W_\mu-$closed nonnegative cone, which is convex along interpolation through any plan respecting the fibers.

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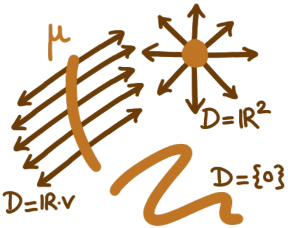
$$\xi \in A \qquad \Longleftrightarrow \qquad [\xi \text{ is centred and } v \in D(x) \text{ for } \xi - \text{almost any } (x, v).]$$



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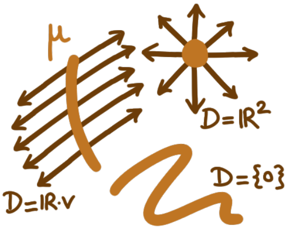

- Proved by passing to the orthogonal complement and exploiting the geometry induced by $\langle \cdot, \cdot \rangle_\mu$.

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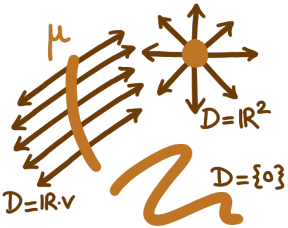

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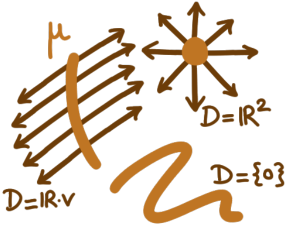

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- Starts with a "nonnegative cone", ends with a "two-sided cone".
- Proves convexity as measures.

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Centred tangent fields

Definition – Geometric tangent cone [Gig08][1] $\mathbf{Tan}_\mu$ is the $W_\mu-$closure of the measure fields of the form $(\pi_x, \lambda\pi_v)_\#\xi$, where $\lambda \geqslant 0$, and ξ induces a geodesic, i.e.

$$\xi = (\pi_x, \pi_y - \pi_x)_\#\eta \qquad \text{for } \eta \text{ optimal plan between } \mu \text{ and some } \nu \in \mathscr{P}_2(\Omega).$$

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Denote $\mathbf{Tan}^0_\mu = \mathbf{Tan}_\mu \cap \mathscr{P}_2(\mathrm{T}\,\Omega)^0_\mu$ the set of centred tangent measure fields.

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By Proposition 4.25 of the same reference, $\mathbf{Tan}_\mu^0$ is a closed convex cone of centred fields.

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By Proposition 4.25 of the same reference, $\mathbf{Tan}^0_\mu$ is a closed convex cone of centred fields.

Corollary Any $\mu \in \mathscr{P}_2(\Omega)$ admits D such that $\xi \in \mathbf{Tan}^0_\mu$ iff ξ is centred and $\xi(\text{graph } D) = 1$.

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Examples

Example 1. If $\mu = \delta_0$, any plan is optimal, so that $\mathbf{Tan}_\mu^0 = \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu^0$, and $D \equiv \mathbb{R}^d$.

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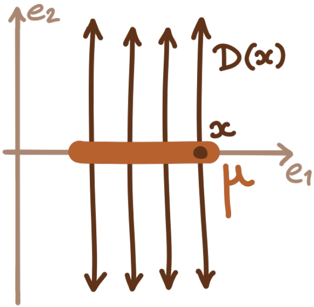

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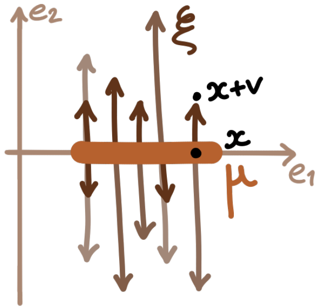

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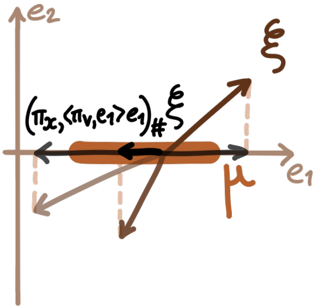

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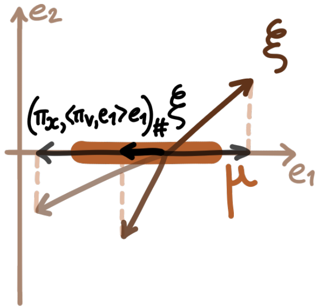

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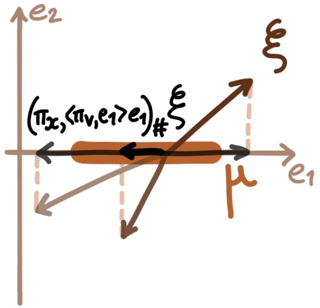
$\simeq$ 1D optimal plan from $\mathcal{L}_{[0,1]}$, hence induced by a map, hence 0, so $v \perp e_1$ ξ−a.e.. Up to details, passes to $\mathbf{Tan}_\mu^0$.



Lott's result

Theorem 1.1 of [Lot16][1] If

- $\mathcal{M}$ is a smooth submanifold of dimension k,
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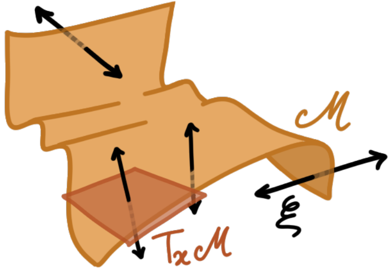
then $\xi \in \mathbf{Tan}^0_\mu$ if and only if (ξ is centred and)

$$v \perp \mathrm{T}_x \mathcal{M} \qquad \xi - \text{almost everywhere.}$$



In other words, $D(x) = (\mathrm{T}_x \mathcal{M})^\perp$.

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Statement

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$$A = \left\{ (x_1, \cdots, x_k, \Phi(x_1, \cdots, x_k)) \;\middle|\; \Phi : \mathbb{R}^k \to \mathbb{R}^{d-k}, \text{ with each } \Phi_i = \text{convex} - \text{convex} \right\}.$$

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[1] L. Zajíček, "On the differentiation of convex functions in finite and infinite dimensional spaces" (1979).
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Theorem 1 of [Zaj79][1] If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is convex, then each $J_k(\varphi)$ is $\sigma-$DC$_\mathsf{k}$.
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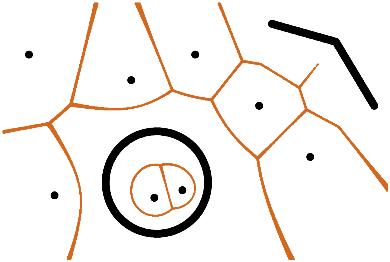
An application

Let $S \subset \mathbb{R}^d$ be closed, and consider

$$T := \left\{ x \in \mathbb{R}^d \;\middle|\; \mathrm{proj}_S(x) \text{ has more than one element} \right\}.$$

Then T is $\sigma-\mathrm{DC_{d\text{-}1}}$.



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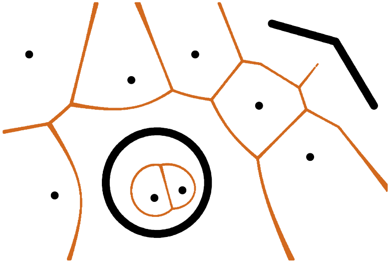

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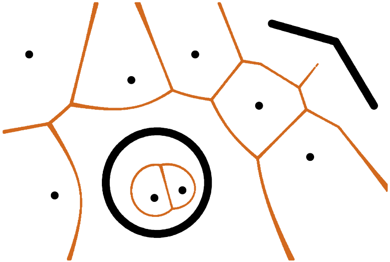

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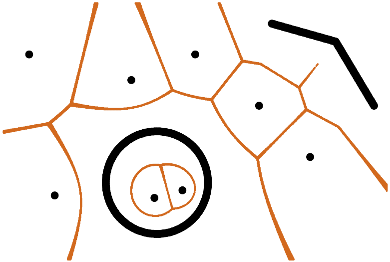

The set on which there is a decision to make is $\sigma-\mathrm{DC}_{\text{d-1}}$.

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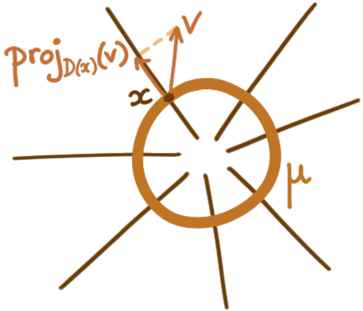
Explicitly, $\xi \in \mathbf{Tan}_\mu^0$ if and only if ξ is (centred and) concentrated on the normal spaces to each A_k.

Projection on $\mathbf{Tan}_{\mu}^{0}$

For each x, denote $\mathrm{proj}_{D(x)} : \mathbb{R}^d \to \mathbb{R}^d$ the projection over $D(x)$.



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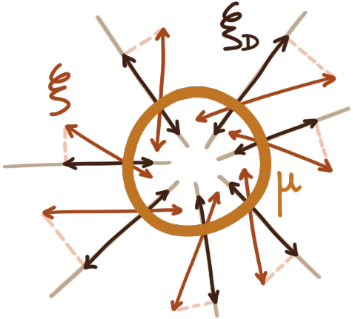

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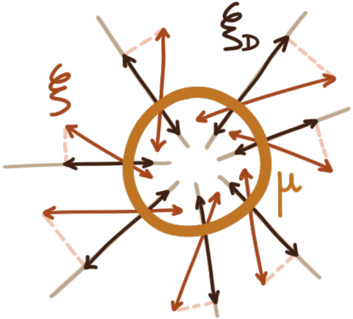

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By construction, $\xi_D \in \mathbf{Tan}_\mu^0$. Conversely, let $\zeta \in \mathbf{Tan}_\mu^0$, and α realize $W_\mu(\zeta, \xi)$. Then

$$W_\mu^2(\zeta, \xi) = \int |v - w|^2 d\alpha \geqslant \int |\mathrm{proj}_{D(x)}(w) - w|^2 d\alpha \geqslant W_\mu^2(\xi_D, \xi).$$

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Kantorovich potentials are semiconvex, so by Zajíček, μ_k is concentrated on a $\sigma-\mathrm{DC_k}$ set A_k.

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Interchanging i and 0, we get $g_i(x) - g_0(x) \perp \mathrm{T}_x J_k(\varphi)$.

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