Locality of the centred Wasserstein tangent cone

Averil Aussedat

February 24, 2026
SNS-Unipi geometric analysis seminar
Notes for a blackboard talk

Università di Pisa

Topic:

- There is an abstract definition of tangent cone to curved spaces.
- In the case of $\mathscr{P}_2$, one observes a certain geometric behaviour on examples.
- This talks proves that this property holds.

Notations:

- $\mathscr{P}_2(\Omega)$ probability measures μ with $\int |x|^2 d\mu < \infty$;
- $\mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ "measure fields", i.e. probabilities on (x,v) with $x \in \Omega$ a point, $v \in \mathrm{T}_x\,\Omega$ a vector. Subscript μ to say that $\pi_{x\#}\xi = \mu$.
- transport plans: $\Gamma(\mu,\nu)$. Then we can consider "optimal plans": the ones minimizing $\int_{(x,y)} |x-y|^2 d\alpha$ among all competitors with the same marginals.
- transport plans within $\mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$; $\Gamma_\mu(\xi,\zeta)$, measures over (x,v,w). Similarly, we can define

$$W_\mu^2(\xi,\zeta) := \inf_{\alpha\in\Gamma_\mu(\xi,\zeta)} \int_{(x,v,w)} |v-w|^2 d\alpha.$$

Definition 1 – Geometric tangent cone Let $\mu \in \mathscr{P}_2(\Omega)$. Construct the geometric tangent cone by

- taking geodesics: collect all $\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ such that $(\pi_x, \pi_x + \pi_v)_\# \xi$ is optimal.
- taking rescalings: collect all $(\pi_x, \lambda \pi_v)_\# \xi$ for $\lambda > 0$ and ξ as before.
- taking the closure: denote $\mathbf{Tan}_\mu$ as the set of $W_\mu-$limits of previous elements.

In words, closed cone over geodesics.

To simplify the statement, we only care about the *centred* elements ξ of $\mathbf{Tan}_\mu$, i.e. such that

$$\int_{v \in \mathrm{T}_x\,\Omega} v d\xi_x(v) = 0.$$

Denote $\mathbf{Tan}_\mu^0$ the set of these elements. By "Gigli's algebra", $\mathbf{Tan}_\mu^0 = (\pi_x, \pi_v - \text{bary}(\pi_x))_\# \mathbf{Tan}_\mu$.

The property that we want to coin can already be seen on simple examples.

- If $\mu = \delta_0$, then any plan is geodesic, and $\mathbf{Tan}^0_\mu = \mathscr{P}_2(\mathrm{T}\,\Omega)^0$ all centred measure fields.
- If $\mu \ll \mathcal{L}^d$, then Brenier-McCann says that $\mathbf{Tan}_\mu = \mathrm{Tan}_\mu$ identifies with the $L^2_\mu-$closure of gradients of smooth, compactly supported functions. Hence $\mathbf{Tan}^0_\mu = \{(id, 0)_\# \mu\}$.
- In between, what appears? Consider $\mu = \mathcal{H}^1 \llcorner [0,1] \times \{0\}$.

We claim that $\xi \in \mathbf{Tan}^0_\mu$ if and only if $v \perp e_1$ for $\xi-$almost any (x, v). One direction: any ξ concentrated on vertical arrows induces a geodesic, because for any $\alpha \in \Gamma(\mu, (\pi_x + \pi_v)_\# \xi)$,

$$\int |x-y|^2 d\alpha \geqslant \int |\mathsf{proj}_{\mathsf{supp}\mu}(y) - y|^2 d\alpha = \int |\mathsf{proj}_{\mathsf{supp}\mu}(y) - y|^2 d\left[(\pi_x + \pi_v)_\# \xi\right]$$
$$= \int |\mathsf{proj}_{\mathsf{supp}\mu}(x+v) - x + v|^2 d\xi = \int |v|^2 d\xi.$$

Converse direction: assume that $\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu^0$ is tangent. First, we take ξ to be optimal. Then, projecting on the horizontal axis keeps it optimal, in that $\zeta := (\pi_x, \langle \pi_v, e_1 \rangle\, e_1)_\# \xi$ is also optimal. Indeed, we check the optimality conditions: for $(x_i, v_i)_{i=1}^N \subset \operatorname{supp} \zeta$, construct $(x_i, v_i + w_i)_{i=1}^N \subset \operatorname{supp} \xi$ with $w_i \perp e_1$. We have

$$\sum_{i=1}^N \langle x_i, x_i + v_i + w_i \rangle \geqslant \sum_{i=1}^N \langle x_{i+1}, x_i + v_i + w_i \rangle$$

$$\sum_{i=1}^N \langle x_i - x_{i+1}, x_i + v_i + w_i \rangle = \sum_{i=1}^N \langle x_i - x_{i+1}, x_i + v_i \rangle \geqslant 0.$$

Hence the optimality of the projection. Now, this identifies with a 1D optimal plan with base measure $\mathcal{L}_{[0,1]}$, hence induced by a map, hence 0 because centred, so $v \perp e_1$ $\xi-$a.e.. This is stable by rescalings and $W_\mu-$limits.

So we observe that centred tangent measure fields are exactly characterized by the fact that they put mass on certain directions. This was observed by Lott[1] in the case of the Hausdorff measure supported on a compact smooth manifold. We can generalize as follows:

Theorem 1 Let $\mu \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$. There exists a measurable set-valued application $D : \Omega \rightrightarrows \mathbb{R}^d$ such that $D(x)$ is a vector subspace for $\mu-$a.e. x, and

$$\xi \in \mathbf{Tan}^0_\mu \qquad \Longleftrightarrow \qquad [\xi \text{ is centred and } v \in D(x) \text{ for } \xi - \text{a.e. } (x,v).]$$

- Actually comes from a more general statement on closed convex cones of centred measure fields.
- False, blatantly false if the "centred" assumption is removed.

[1] J. Lott. "On tangent cones in Wasserstein space". In: *Proceedings of the American Mathematical Society* 145.7 (Dec. 2016), pp. 3127–3136

Idea of the proof: by Gigli's results, $\mathbf{Tan}^0_\mu$ is a $W_\mu-$closed (obvious) convex cone (obvious) of centred (obvious) measure fields, where "convex" means that whenever $\xi, \zeta \in \mathbf{Tan}^0_\mu$ and $\alpha \in \Gamma_\mu(\xi, \zeta)$, then

$$(\pi_x, (1-t)\pi_v + t\pi_w)_\# \alpha \in \mathbf{Tan}^0_\mu .$$

Theorem: it is equivalent that $A \subset \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ is a closed convex cone of centred measure field, and that there exists D describing it as before.

- One direction is obvious.
- Conversely, one constructs D by taking spans of countably many particular elements of A, and shows that any other element must be concentrated on this span.

We can go (way) further in describing the set-valued map D. Let us introduce a particular class of sets.

Definition 2 – $\sigma-$DC$_k$ sets Let $k \in [\![0, d]\!]$. A set $A \subset \mathbb{R}^d$ is DC$_k$ (for Difference of Convex functions of dimension k) if, up to a permutation of variables, it can be written as

$$A = \left\{(x_1, \cdots, x_k, \Phi(x_1, \cdots, x_k)) \;\middle|\; \Phi : \mathbb{R}^k \to \mathbb{R}^{d-k}, \text{ and each entry is convex - convex.}\right\}$$

A set A is $\sigma-$DC$_k$ if is can be covered by countably many DC$_k$ sets.

$\sigma-$DC$_0$ sets are exactly countable sets; the only DC$_d$ set is $\mathbb{R}^d$.
These sets are called c–c convex hypersurfaces by Zajíček[2], and $\delta-$convex surfaces by Pavlica[3]. One should also cite Alberti[4] for hidden rediscovery.

[2] L. Zajíček. "On the differentiation of convex functions in finite and infinite dimensional spaces". In: *Czechoslovak Mathematical Journal* 29.3 (1979), pp. 340–348

[3] D. Pavlica. "On the points of non-differentiability of convex functions". In: *Commentationes Mathematicae Universitatis Carolinae* 45.4 (2004), pp. 727–734

[4] G. Alberti. "On the structure of singular sets of convex functions". In: *Calculus of Variations and Partial Differential Equations* 2.1 (Jan. 1994), pp. 17–27

Now we can be more precise about D.

Theorem 2 Let $\mu \in \mathscr{P}_2(\Omega)$. Then there exists a unique decomposition $\mu = \mu_0 + \mu_1 + \cdots + \mu_d$ into mutually singular measures, where

(a) each μ_k is concentrated on a $\sigma-\mathsf{DC_k}$ set A_k, and gives 0 mass to $\sigma-\mathsf{DC_j}$ sets for $j < k$,

(b) the set A_k admits a tangent space μ_k−almost everywhere (in a sense to be precised, but mostly what you think),

(c) there holds $D(x) = (\mathrm{T}_x A_k)^\perp$ for μ_k−almost any $x \in \Omega$. In particular, $\dim D(x) = d - k$ on μ_k.

Some ideas of why (a) holds:

- construct $(\mu_k)_k$ by restricting μ to the sets where $\dim D = d - k$. So mutually singular.
- Then, by definition, one can construct a centred tangent measure field ξ_k attached to μ_k that shoots mass in $d - k$ independent directions.
- Because ξ_k is tangent, it is almost optimal. Therefore, it is almost valued in the subdifferential of a semiconvex function.
- **By Zajíčék's theorem,** the set on which a semiconvex function can admit $d - k$ independent directions is $\sigma - \mathrm{DC_k}$, and this is an equivalence. Hence μ_k must sit on a $\sigma - \mathrm{DC_k}$ set.
- If μ_k was to give mass to some $\sigma - \mathrm{DC_j}$ set of $j < k$, we could construct a convex function admitting $d - j > d - k$ independent directions in the subdifferential, hence an optimal plan concentrated on these directions, hence D must be of larger dimension: absurd.

Some ideas of why (b) holds:

- the definition of "A_k has a tangent space at x" is the following: cover A_k by countably many graphs-up-to-permutations of DC functions $(\varphi_{k+1} - \psi_{k+1}, \cdots, \varphi_d - \psi_d)$ from $\mathbb{R}^k$ to $\mathbb{R}^{d-k}$. Then, if x belongs to one of the graphs, all φ_i, ψ_i have to be differentiable at (the antecedent of) x and their derivatives must span the same vector subspace P, declared to be $\mathrm{T}_x A_k$.
- Two obstructions to the existence of $\mathrm{T}_x A_k$;
  - some φ_i, ψ_i could not be differentiable. But these are convex functions from $\mathbb{R}^k$, so the singular set is DC_{k-1}, and injects back in $\mathbb{R}^d$ as a DC_{k-1} set, *not seen by* μ_k.
  - the tangents provided by two graphs could intersect transversely. Trick: represent each graph as the singular set of a convex function, and sum them. Transverse intersection means that the sum has a subdifferential *of one more dimension* at the point, so that transverse intersections are $\sigma-\mathrm{DC}_{\text{k-1}}$, not seen by μ_k.

So, $\mathrm{T}_x A_k$ exists μ_k-almost everywhere.

Some ideas why (c) holds: working with Kantorovich potentials, this boils down to proving that sets of singularities of convex functions and spans of their subdifferentials are orthogonal *in good cases*.

Magic Lemma Let φ be convex with $J_k(\varphi) = \{x \mid \dim \partial_x \varphi \geqslant d-k\}$ a smooth surface. Assume that for any $x \in J_k(\varphi)$, there holds $\partial_x \varphi = \text{conv}\,\{g_0(x), \cdots, g_{d-k}(x)\}$ for continuous $(g_i)_i$. Then

$$\text{span}\,\partial_x \varphi := \text{span}\,\{g_i(x) - g_0(x)\}_{i=1}^{d-k} \quad \perp \quad \mathrm{T}_x\, J_k(\varphi).$$

For any smooth curve $\gamma \subset J_k(\varphi)$, there holds

$$\varphi(\gamma_t) \geqslant \varphi(\gamma_0) + \langle g_i(\gamma_0), \gamma_t - \gamma_0 \rangle \geqslant \varphi(\gamma_t) + \langle g_0(\gamma_t), \gamma_0 - \gamma_t \rangle + \langle g_i(\gamma_0), \gamma_t - \gamma_0 \rangle \,.$$

Therefore, dividing by $t > 0$,

$$0 \geqslant \left\langle g_i(\gamma_0) - g_0(\gamma_t), \frac{\gamma_t - \gamma_0}{t} \right\rangle \underset{t \searrow 0}{\longrightarrow} \langle g_i(\gamma_0) - g_0(\gamma_0), \dot{\gamma}_0 \rangle \,.$$

Interchanging i and 0, we get $g_i(x) - g_0(x) \perp \mathrm{T}_x\, J_k(\varphi)$.

Additional comments:

- The spaces $D(x)$ are "normal" to the measure μ, but characterize "tangent" measure fields, and there is no contradiction.
- The tangent spaces A_k could be thought as a "tangent space to μ". Many definitions exist:
  - This is for sure not the same as Preiss's tangent measures, although (theorem) the support of any Preiss tangent measure to μ_k is contained in $\mathrm{T}_x A_k$ μ_k almost everywhere.
  - This is not the same either as the Bouchitté-Alberti-Marchese-Buttazzo-Champion-Jimenez tangent space, which is related to $\mathcal{C}^1$/rectifiable surfaces as the present one is to $\mathcal{C}^2/\sigma-$DC surfaces. However, here as well, the BAMBCJ tangent space is always included in $\mathrm{T}_x A_k$ μ_k-almost everywhere.
- The result can provide information on $\mathbf{Tan}_\mu$ (removing the assumption that the barycenter is 0); first, $\xi \in \mathbf{Tan}_\mu$ must have a centred part in $\mathbf{Tan}_\mu^0$, to which the theorem applies. Moreover, it can be shown that the barycenter b has a projection on D which is not constrained, i.e. one can add any element of D to b and still obtain the barycenter of a tangent field. Conclusion, as Lott puts it: *the barycenter of a tangent element is only constrained through its "tangential" component,* here the projection on $D^\perp = \mathrm{T}_x A_k$.