Measures are fun

Introduction to the Wasserstein distance and its geometry

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LITIS doctoral seminar

LMI - EA 3226

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What is a measure

An object modelling the presence of mass.

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Def 1 – Probability measure
Let $(\Omega; \Sigma)$ be a measurable space. A probability measure μ is an application from Σ to $[0,1]$ such that $\mu(\Omega) = 1$ and for any disjoint sequence $(A_i)_{i\in\mathbb{N}}$,

$$\mu\left(\bigsqcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

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How to compute the distance between to measures?

With the total variation

$$|\mu - \nu|_{\mathsf{TV}} = \sup_{\substack{\mathcal{A} \in \text{ countable measurable} \\ \text{partitions of } \Omega}} \sum_{A \in \mathcal{A}} |\mu(A) - \nu(A)|,$$

we would have $|\delta_0 - \delta_t|_{\mathsf{TV}} = 2$ regardless of how close t is to 0.

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we would have $|\delta_0 - \delta_t|_{\mathsf{TV}} = 2$ regardless of how close t is to 0. **Not physical.**

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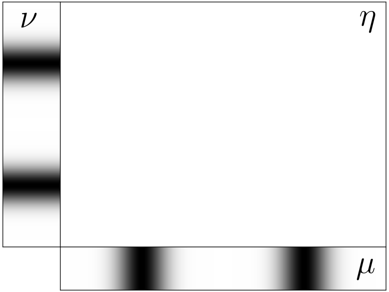
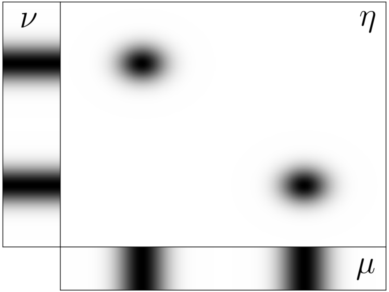


Introducing the Wasserstein distance

Let $\mu, \nu \in \mathscr{P}(\Omega)$ be two probability measures, and denote the set of transport plans by

$$\Gamma(\mu,\nu) := \left\{\eta \in \mathscr{P}((\Omega)^2) \;\middle|\; \pi_x\#\eta = \mu,\ \pi_y\#\eta = \nu\right\},$$



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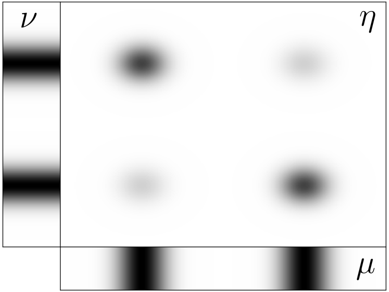

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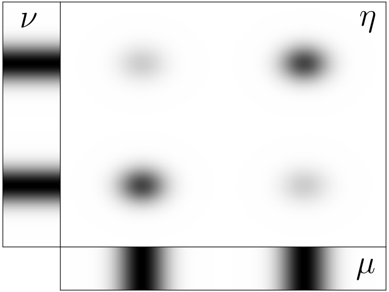

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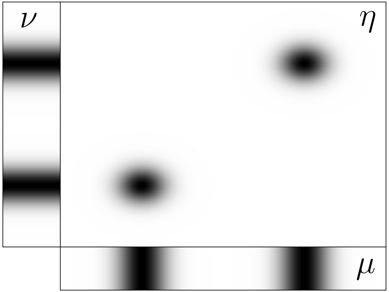

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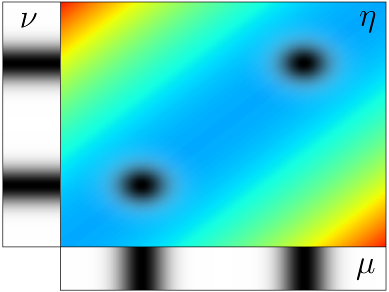
the squared Wasserstein distance by

$$d^2_{\mathcal{W}}(\mu,\nu) := \inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^2 \, d\eta(x,y).$$



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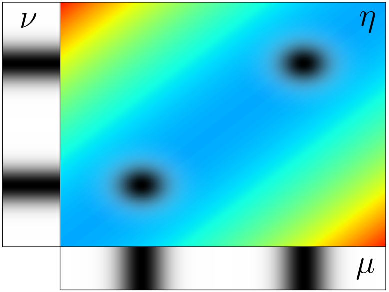

Def 2 – [San15] We call **Wasserstein space** the set $\mathscr{P}_2(\Omega)$ or measures μ such that $d_{\mathcal{W}}(\mu,\delta_0)$ is finite, endowed with the distance $d_{\mathcal{W}}$.

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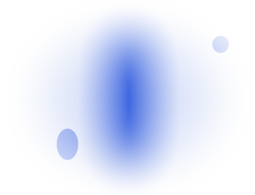

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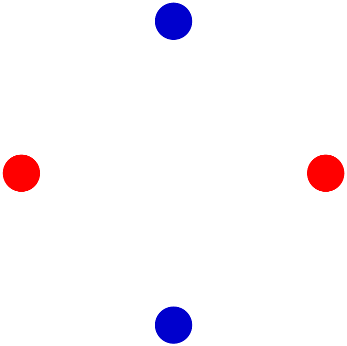

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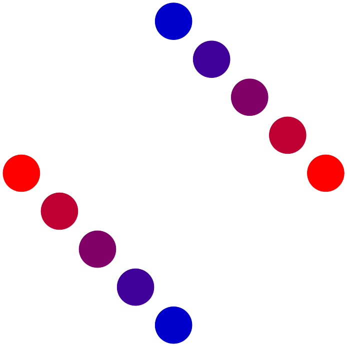

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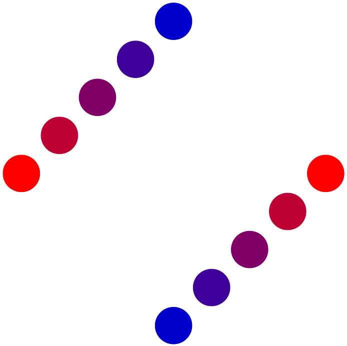

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Application: continuity equations

Def 3 – Fokker-Planck equation Given an initial measure $\nu \in \mathscr{P}_2(\Omega)$, find a curve $(\mu_t)_{t\in[0,T]}$ that satisfies $\mu_0 = \nu$ and solves in the sense of distributions

$$\partial_t \mu_t + \text{div}\,(f(t,x,\mu_t)\#\mu_t) = 0 \quad \forall t \in (0,T). \tag{1}$$

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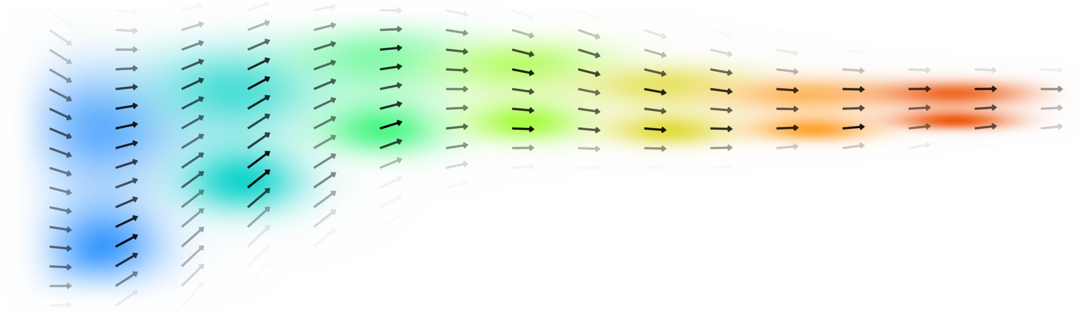

The Cauchy-Lipschitz framework

Theorem 1 – Well-posedness [BF23] Assume that $f : [0,T] \times \Omega \times \mathscr{P}_2(\Omega) \to \mathbb{R}^d$ is bounded and Lipschitz-continuous in all its variables (with respect to $d_{\mathcal{W}}$ for the measure variable). Then there exists an unique solution to (1), that is given by

$$\mu_t = \Phi_t^\mu \# \nu, \quad \text{where} \quad \frac{d}{dt}\Phi_t^\mu(x) = f\left(t, \Phi_t^\mu(x), \mu_t\right) \quad \text{and} \quad \Phi_0^\mu(x) = x.$$

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For instance,

$$f(t,x,\mu) := g(t,x) + \int_{y\in\Omega} \varphi(t,x,y)d\mu(y)$$

where $g : [0,T] \times \Omega \to \mathbb{R}^d$ and $\varphi \in \mathcal{C}_c([0,T] \times \Omega \times \Omega; \mathbb{R}^d)$ are Lipschitz in all variables.

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Consider the vector field f given by $f(x) = 1$ is $x < 0$, and $f(x) = 0$ if $x \geqslant 0$.

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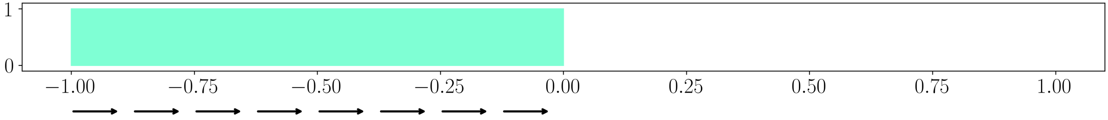
$$\mu_t = \mathcal{L}_{[(-1+t)_+,0]} + \min(t,1)\delta_0.$$



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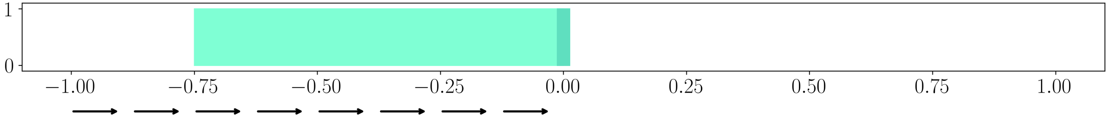

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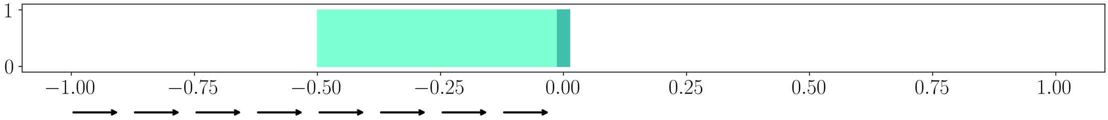

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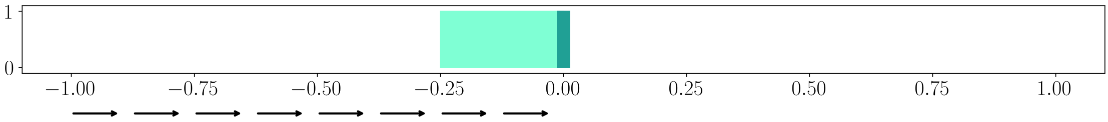

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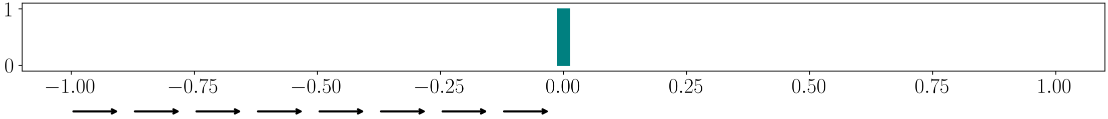

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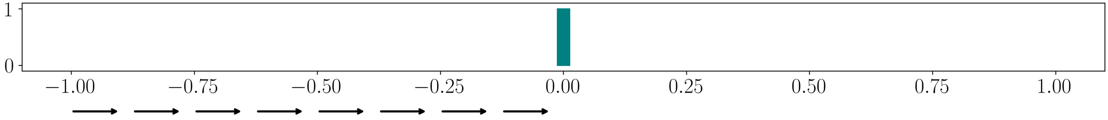

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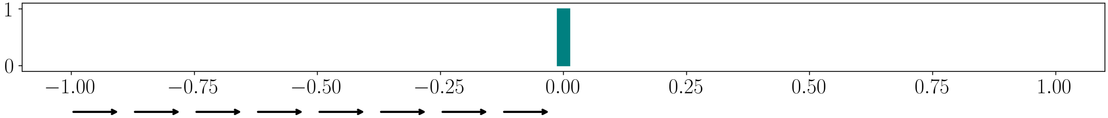

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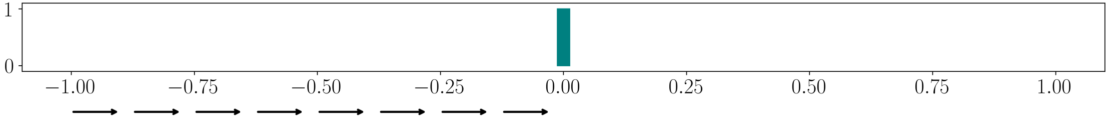

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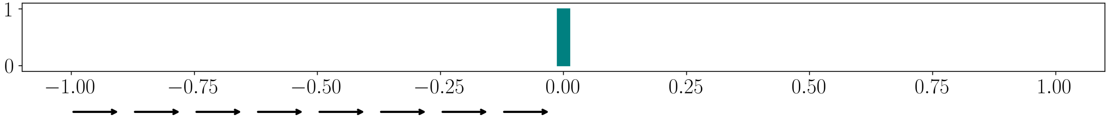

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$$\int_{x\in\mathbb{R}} \langle \nabla\varphi(x), f(x)\rangle \, d\mu_t(x) = \int_{x=(-1+t)_+}^{0} \varphi'(x)dx = \begin{cases} \varphi(0) - \varphi(-1+t) & \text{if } t \in [0,1), \\ 0 & t \geqslant 1. \end{cases}$$

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- Here f is not Lipschitz-continuous, actually no available theory in this case.

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The Hodge-Helmholtz decomposition

Theorem 2 – HH decomposition [Lad87] Let $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. There exists two uniquely defined vector fields $g, h \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ such that

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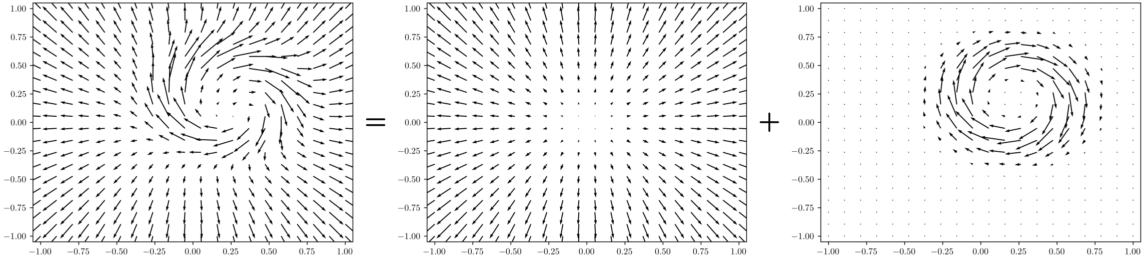
Theorem 2 – HH decomposition [Lad87] Let $f \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. There exists two uniquely defined vector fields $g, h \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ such that

$$f = g + h, \qquad g \simeq \nabla\varphi, \qquad \text{div}\,(h) \simeq 0. \tag{2}$$



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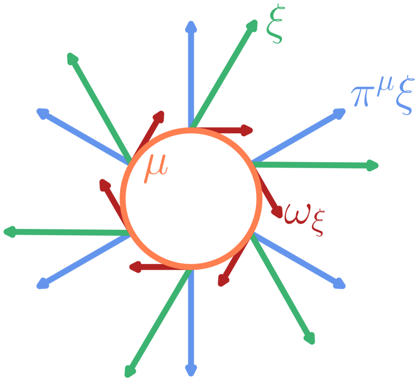
Theorem 3 – Hodge decomposition Any field $\xi \in \mathscr{P}_2(\mathrm{T}\Omega)$ decomposes in a "tangent" component akin to a gradient, and a "divergence-free" component.



Thank you!

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