Swirling measures

The quotient structure of the tangent cone to the Wasserstein space

Averil Prost
LMI, INSA Rouen Normandie
Hasnaa Zidani and Nicolas Forcadel (LMI)

July 5, 2024
Journée de la fédération Normandie Mathématiques, Rouen

Helmholtz decomposition

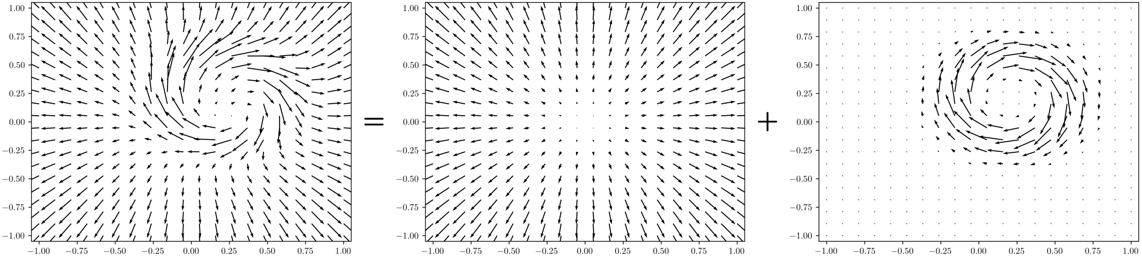
Theorem – HH decomposition [Lad87] Let $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. There exists two uniquely defined vector fields $g, h \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$f = g + h, \qquad g \in \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2}, \qquad h \in \overline{\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{div}\varphi = 0\}}^{L^2}.$$

Helmholtz decomposition

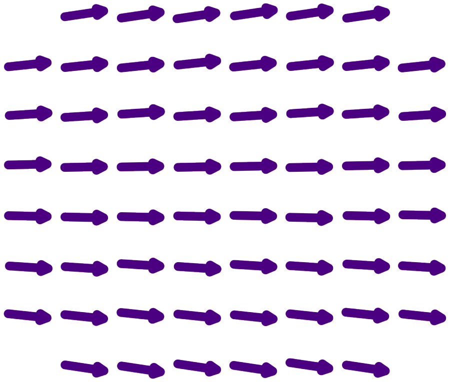
Theorem – HH decomposition [Lad87] Let $f \in L^2(\mathbb{R}^d;\mathbb{R}^d)$. There exists two uniquely defined vector fields $g, h \in L^2(\mathbb{R}^d;\mathbb{R}^d)$ such that

$$f = g + h, \qquad g \in \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2}, \qquad h \in \overline{\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{div}\varphi = 0\}}^{L^2}.$$



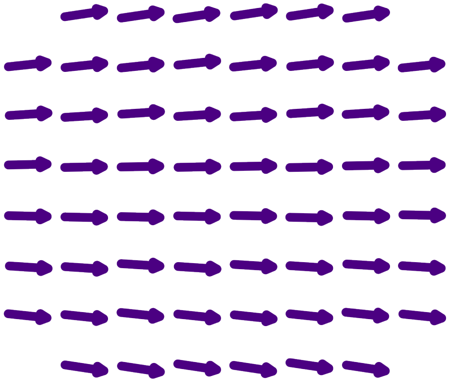
Aim of the talk

- Introduce a generalization in the case of measure fields.



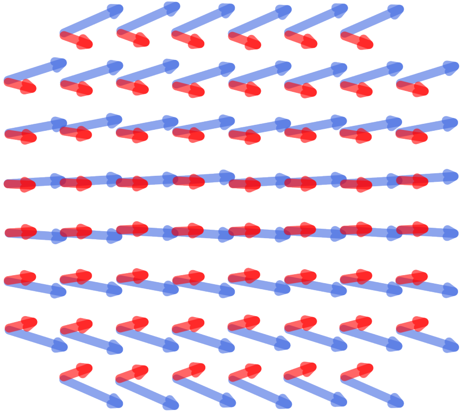
Aim of the talk

- Introduce a generalization in the case of measure fields.
    - Vector field : $x \mapsto f(x) \in \mathsf{T}_x\mathbb{R}^d$.



Aim of the talk

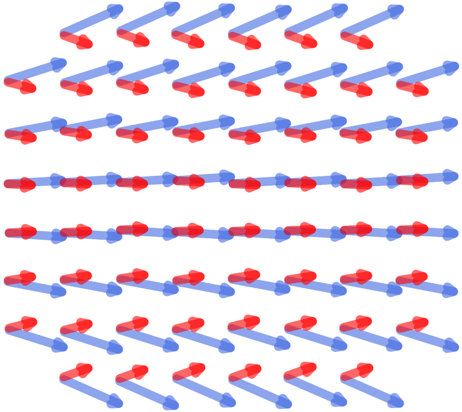
- Introduce a generalization in the case of measure fields.
    - Vector field : $x \mapsto f(x) \in \mathrm{T}_x\mathbb{R}^d$.
    - Measure field : $x \mapsto \xi_x \in \mathscr{P}(\mathrm{T}_x\mathbb{R}^d)$.



Aim of the talk

- Introduce a generalization in the case of measure fields.
    - Vector field : $x \mapsto f(x) \in \mathrm{T}_x\mathbb{R}^d$.
    - Measure field : $x \mapsto \xi_x \in \mathscr{P}(\mathrm{T}_x\mathbb{R}^d)$.
    - To allow for definitions only $\mu-$a.e., better suited to consider measures on $\mathrm{T}\mathbb{R}^d = \{(x, v)\}$ with $x \in \mathbb{R}^d$ and $v \in \mathrm{T}_x\mathbb{R}^d$, i.e.

$$\xi \in \mathscr{P}(\mathrm{T}\mathbb{R}^d).$$



Aim of the talk

- Introduce a generalization in the case of measure fields.
    - Vector field : $x \mapsto f(x) \in \mathrm{T}_x\mathbb{R}^d$.
    - Measure field : $x \mapsto \xi_x \in \mathscr{P}(\mathrm{T}_x\mathbb{R}^d)$.
    - To allow for definitions only $\mu-$a.e., better suited to consider measures on $\mathrm{T}\mathbb{R}^d = \{(x, v)\}$ with $x \in \mathbb{R}^d$ and $v \in \mathrm{T}_x\mathbb{R}^d$, i.e.

$$\xi \in \mathscr{P}(\mathrm{T}\mathbb{R}^d).$$

- Derive a formulation of the tangent cone to the Wasserstein space.

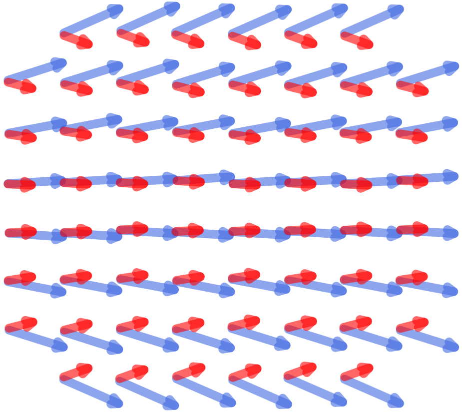


Table of Contents

Wasserstein space

Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Wasserstein space

Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Def – Wasserstein distance

$$d_{\mathcal{W}}^2(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$

$\left(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}}\right)$ is a complete geodesic metric space.

Wasserstein space

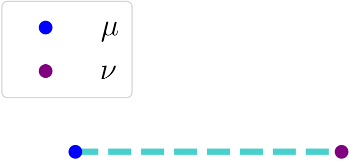
Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Def – Wasserstein distance

$$d^2_{\mathcal{W}}(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$



$(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}})$ is a complete geodesic metric space.

Wasserstein space

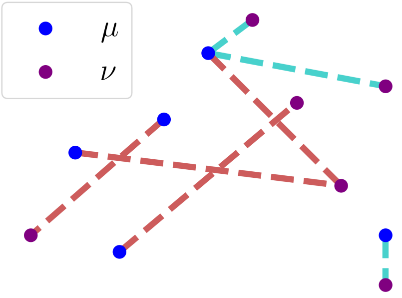
Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Def – Wasserstein distance

$$d_{\mathcal{W}}^2(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$



$\left(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}}\right)$ is a complete geodesic metric space.

Wasserstein space

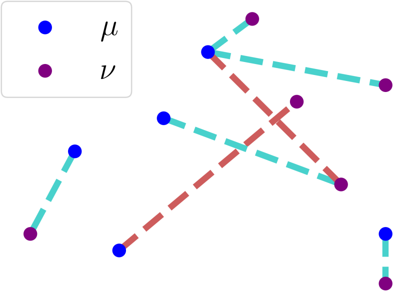
Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Def – Wasserstein distance

$$d^2_{\mathcal{W}}(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$



$(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}})$ is a complete geodesic metric space.

Wasserstein space

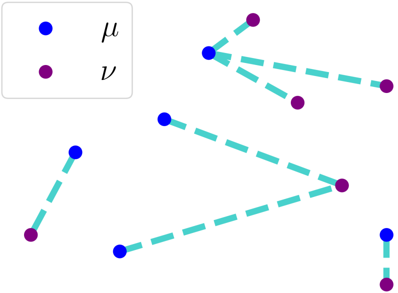
Let μ_0, μ_1 be Borel probability measures such that $\int_{x\in\mathbb{R}^d} |x|^2 \, d\mu_i(x) < \infty$.

Def – Wasserstein distance

$$d^2_{\mathcal{W}}(\mu,\nu) := \inf \int_{(x,y)\in(\mathbb{R}^d)^2} |x-y|^2 \, d\omega,$$

where $\omega \in \mathscr{P}_2((\mathbb{R}^d)^2)$ is such that

$$\int_x d\omega = \nu \quad \text{and} \quad \int_y d\omega = \mu.$$



$(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}})$ is a complete geodesic metric space.

Measure fields

Let $\mathsf{T}\mathbb{R}^d := \{(x,v) \mid x \in \mathbb{R}^d,\ v \in \mathsf{T}_x\mathbb{R}^d\}$. Denote $f\#\alpha$ the measure $\alpha(f^{-1}(\cdot))$.

Measure fields

Let $\mathrm{T}\mathbb{R}^d := \left\{(x,v) \mid x \in \mathbb{R}^d,\ v \in \mathrm{T}_x\mathbb{R}^d\right\}$. Denote $f\#\alpha$ the measure $\alpha(f^{-1}(\cdot))$.

Def – Measure field A measure $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$ is a measure field attached to $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, denoted $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, if $\pi_x\#\xi = \mu$.

Measure fields

Let $\mathrm{T}\mathbb{R}^d := \left\{(x, v) \mid x \in \mathbb{R}^d,\ v \in \mathrm{T}_x\mathbb{R}^d\right\}$. Denote $f\#\alpha$ the measure $\alpha(f^{-1}(\cdot))$.

Def – Measure field A measure $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$ is a measure field attached to $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, denoted $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, if $\pi_x\#\xi = \mu$.

Any vector field $f \in L^2_\mu$ identifies with $\xi = f\#\mu$, for which $\xi_x = \delta_{(x,f(x))}$.

Def – Distance between measure fields [Gig08]

$$W_\mu^2(\xi, \zeta) := \int_{x \in \mathbb{R}^d} d^2_{\mathcal{W}, \mathrm{T}_x\mathbb{R}^d}(\xi_x, \zeta_x)\, d\mu(x).$$

Measure fields

Let $\mathrm{T}\mathbb{R}^d := \left\{(x,v) \mid x \in \mathbb{R}^d,\ v \in \mathrm{T}_x\mathbb{R}^d\right\}$. Denote $f\#\alpha$ the measure $\alpha(f^{-1}(\cdot))$.

Def – Measure field A measure $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$ is a measure field attached to $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, denoted $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, if $\pi_x\#\xi = \mu$.

Any vector field $f \in L^2_\mu$ identifies with $\xi = f\#\mu$, for which $\xi_x = \delta_{(x,f(x))}$.

Def – Distance between measure fields [Gig08]

$$W_\mu^2(\xi,\zeta) := \int_{x\in\mathbb{R}^d} d^2_{\mathcal{W},\mathrm{T}_x\mathbb{R}^d}(\xi_x,\zeta_x)\,d\mu(x).$$

In particular, $W_\mu(f\#\mu, g\#\mu) = \|f-g\|_{L^2_\mu}$.

Tangent cone: the geometric approach

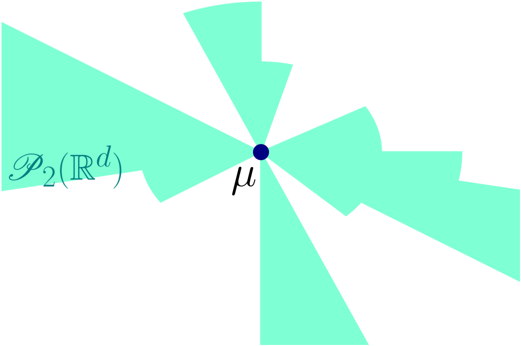
Canonical construction of the tangent cone [AGS05, Gig08]

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$.

Tangent cone: the geometric approach

Canonical construction of the tangent cone [AGS05, Gig08]

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$.



Tangent cone: the geometric approach

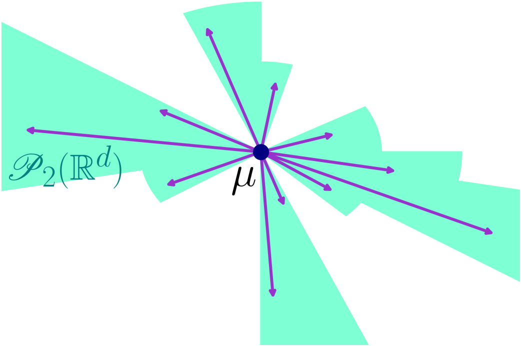
Canonical construction of the tangent cone [AGS05, Gig08]

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\eta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\eta$$

  is a geodesic;



Tangent cone: the geometric approach

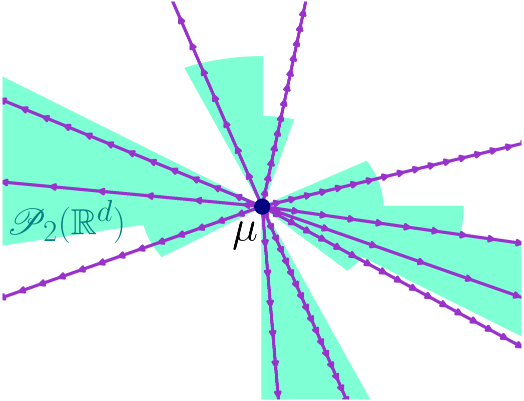
Canonical construction of the tangent cone [AGS05, Gig08]

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\eta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\eta$$

  is a geodesic;
- the positive cone $\alpha \cdot \eta$ for all $\alpha \geqslant 0$,



Tangent cone: the geometric approach

Canonical construction of the tangent cone [AGS05, Gig08]

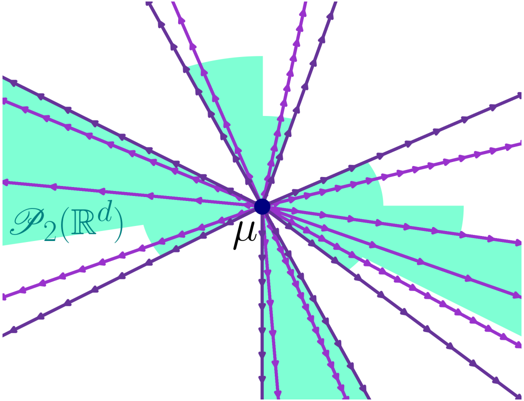
Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\eta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\eta$$

  is a geodesic;
- the positive cone $\alpha \cdot \eta$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.



Link with gradient fields

What is the link between $\mathbf{Tan}_\mu$ and gradient fields?

Link with gradient fields

What is the link between $\mathbf{Tan}_\mu$ and gradient fields? First, if μ is "kind", direct representation.

Theorem – Tangent space to a regular measure [Bre91] Assume that μ is absolutely continuous with respect to the Lebesgue measure. Then

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) = \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu} \# \mu =: \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

Link with gradient fields

What is the link between **Tan**$_\mu$ and gradient fields? First, if μ is "kind", direct representation.

Theorem – Tangent space to a regular measure [Bre91] Assume that μ is absolutely continuous with respect to the Lebesgue measure. Then

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) = \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^\infty\}}^{L^2_\mu} \# \mu =: \mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

In the general case, one has the following.

Theorem – Vertical superposition of the tangent cone For any $\eta \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$, there exists $\varpi \in \mathscr{P}_2(\mathrm{Tan}_\mu)$ such that for all $\varphi \in \mathcal{C}_b(\mathbb{R}^d;\mathbb{R})$,

$$\int_{(x,v)\in \mathrm{T}\mathbb{R}^d} \varphi(x,v)d\eta(x,v) = \int_{b\in \mathrm{Tan}_\mu} \int_{x\in\mathbb{R}^d} \varphi(x,b(x))d\mu(x)d\varpi(b).$$

Table of Contents

Solenoidal measure fields

In L^2_μ, solenoidal (divergence-free) fields are orthogonal to gradient vector fields.

Solenoidal measure fields

In L^2_μ, solenoidal (divergence-free) fields are orthogonal to gradient vector fields.

Def – Metric scalar product Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, and $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$.

$$\langle \xi, \zeta \rangle_\mu := \frac{1}{2}\left[\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi, \zeta) \right], \qquad \text{where } \|\xi\|_\mu := W_\mu^2(\xi, 0_\mu).$$

Solenoidal measure fields

In L^2_μ, solenoidal (divergence-free) fields are orthogonal to gradient vector fields.

Def – Metric scalar product Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$, and $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$.

$$\langle \xi, \zeta \rangle_\mu := \frac{1}{2}\left[\|\xi\|_\mu^2 + \|\zeta\|_\mu^2 - W_\mu^2(\xi, \zeta)\right], \qquad \text{where } \|\xi\|_\mu := W_\mu^2(\xi, 0_\mu).$$

We may now define the set $\mathbf{Sol}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Def – Solenoidal measure fields An element $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ is said solenoidal if

$$\langle \eta, \zeta \rangle_\mu = 0 \qquad \forall \eta \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d).$$

Examples

- If $\mu = \delta_x$, then $\textbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone.

Examples

- If $\mu = \delta_x$, then $\mathbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone. **Very small!**

Examples

- If $\mu = \delta_x$, then $\mathbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone. **Very small!**
- If μ is absolutely continuous, then $\mathbf{Tan}_\mu = \mathrm{Tan}_\mu$ and for any $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$,

$$\langle f\#\mu, \zeta\rangle_\mu = \langle f, \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)\rangle_{L^2_\mu}, \qquad \text{where } \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)(x) = \int_{v\in \mathrm{T}_x\mathbb{R}^d} v d\zeta(x,v).$$

Examples

- If $\mu = \delta_x$, then $\textbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone. **Very small!**
- If μ is absolutely continuous, then $\textbf{Tan}_\mu = \mathrm{Tan}_\mu$ and for any $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$,

$$\langle f\#\mu, \zeta\rangle_\mu = \langle f, \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)\rangle_{L^2_\mu}, \qquad \text{where } \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)(x) = \int_{v\in \mathrm{T}_x\mathbb{R}^d} v d\zeta(x,v).$$

  Hence $\zeta \in \textbf{Sol}_\mu$ iff its barycenter is a solenoidal vector field.

Examples

- If $\mu = \delta_x$, then $\textbf{Sol}_\mu = \{0_\mu\}$. Indeed, any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ induces a geodesic, hence belongs to the tangent cone. **Very small!**
- If μ is absolutely continuous, then $\textbf{Tan}_\mu = \mathrm{Tan}_\mu$ and for any $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$,

$$\langle f\#\mu, \zeta\rangle_\mu = \langle f, \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)\rangle_{L^2_\mu}, \qquad \text{where } \mathrm{Bary}_{\mathrm{T}\mathbb{R}^d}(\zeta)(x) = \int_{v\in\mathrm{T}_x\mathbb{R}^d} v d\zeta(x,v).$$

Hence $\zeta \in \textbf{Sol}_\mu$ iff its barycenter is a solenoidal vector field. **Very large!**

HH decomposition for measure fields

Recall that in the classical case, $f = g + h$, or in weak form,

$$\int_x \varphi(x, f(x))d\mu = \int_x \varphi(x, g(x) + h(x))d\mu \qquad \forall \varphi \in \mathcal{C}_b(\mathsf{T}\mathbb{R}^d; \mathbb{R}).$$

HH decomposition for measure fields

Recall that in the classical case, $f = g + h$, or in weak form,

$$\int_x \varphi(x, f(x))d\mu = \int_x \varphi(x, g(x) + h(x))d\mu \qquad \forall \varphi \in \mathcal{C}_b(\mathsf{T}\mathbb{R}^d; \mathbb{R}).$$

Theorem – HH decomposition For any $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$, there exists an unique pair $\eta \in \mathbf{Tan}_\mu$ and $\zeta \in \mathbf{Sol}_\mu$ such that for some measurable family $(\alpha_x)_x$ with $\alpha_x \in \Gamma(\eta_x, \zeta_x)$ for a.e. $x \in \mathrm{supp}\mu$,

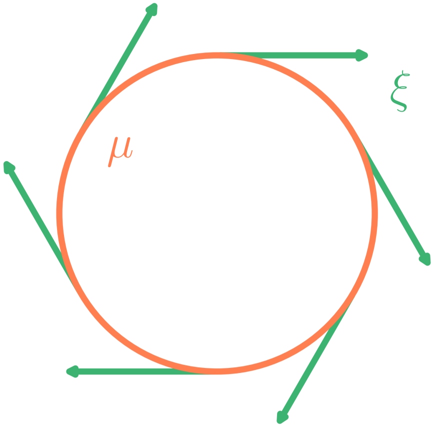
$$\int_{(x,v)} \varphi(x, v)d\xi = \int_{\substack{x\in\mathbb{R}^d,\\ (v,w)\in(\mathsf{T}_x\mathbb{R}^d)^2}} \varphi(x, v + w)d[\alpha_x \otimes \mu] \qquad \forall \varphi \in \mathcal{C}_b(\mathsf{T}\mathbb{R}^d; \mathbb{R}).$$

Table of Contents

Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

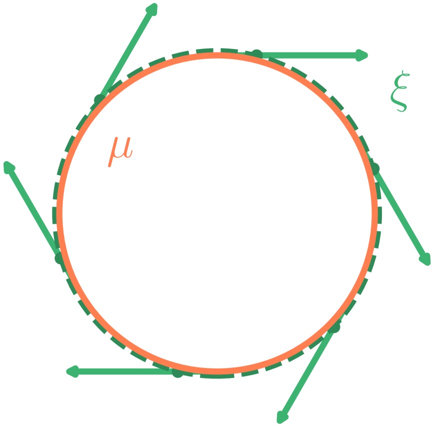
$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

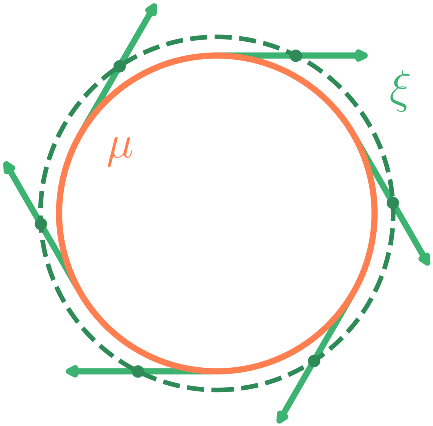
$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

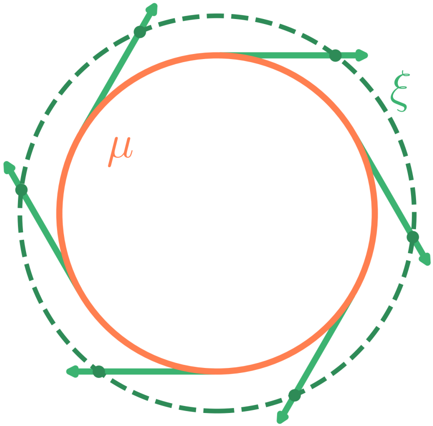
$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

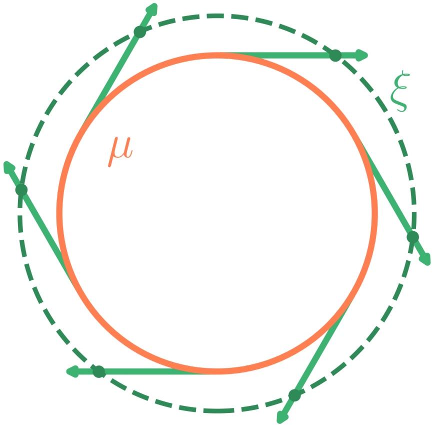


Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

In this case, the tangent component of ξ is 0_μ.



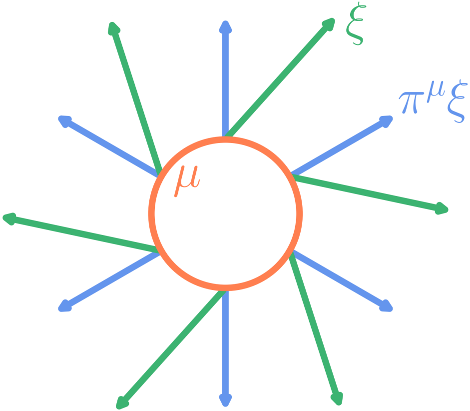
Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

In this case, the tangent component of ξ is 0_μ.
In general, if $\pi^\mu \xi$ denotes the tangent component,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}((\pi_x + h\pi_v)\#\pi^\mu\xi, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



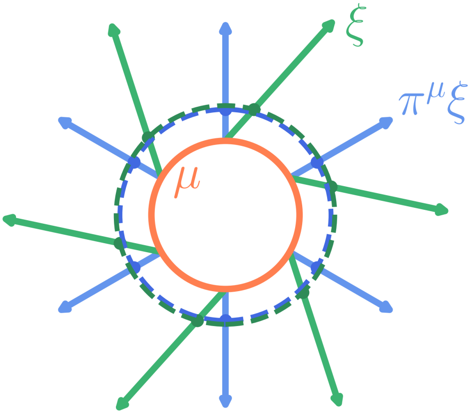
Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h\searrow 0}\frac{d_{\mathcal{W}}(\mu,(\pi_x+h\pi_v)\#\xi)}{h}=0.$$

In this case, the tangent component of ξ is 0_μ.
In general, if $\pi^\mu\xi$ denotes the tangent component,

$$\lim_{h\searrow 0}\frac{d_{\mathcal{W}}((\pi_x+h\pi_v)\#\pi^\mu\xi,(\pi_x+h\pi_v)\#\xi)}{h}=0.$$



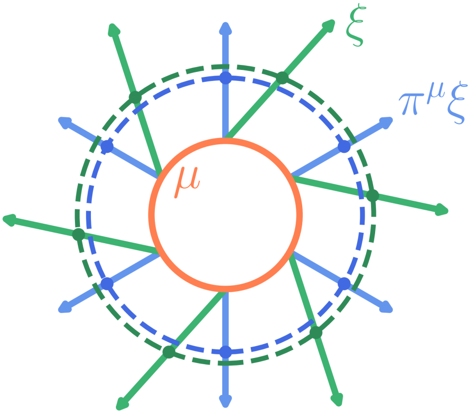
Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

In this case, the tangent component of ξ is 0_μ.
In general, if $\pi^\mu \xi$ denotes the tangent component,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}((\pi_x + h\pi_v)\#\pi^\mu\xi, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



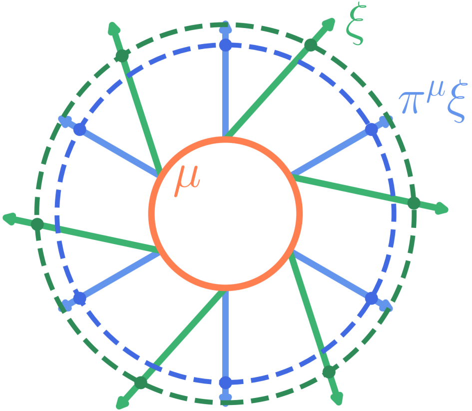
Escaping behaviour

Proposition A measure field ξ is solenoidal if and only if

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}(\mu, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$

In this case, the tangent component of ξ is 0_μ.
In general, if $\pi^\mu \xi$ denotes the tangent component,

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}((\pi_x + h\pi_v)\#\pi^\mu\xi, (\pi_x + h\pi_v)\#\xi)}{h} = 0.$$



Tangent cone: quotient approach

Construction of the tangent cone

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\xi$$

  is a geodesic;
- the positive cone $\alpha \cdot \xi$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Quotient construction

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$.

Tangent cone: quotient approach

Construction of the tangent cone

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\xi$$

  is a geodesic;
- the positive cone $\alpha \cdot \xi$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Quotient construction

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Define $\xi \sim_\mu \zeta$ if

$$d_{\mathcal{W}}\left((\pi_x + h\pi_v)\#\xi, (\pi_x + h\pi_v)\#\zeta\right) = o(h).$$

Tangent cone: quotient approach

Construction of the tangent cone

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\xi$$

  is a geodesic;
- the positive cone $\alpha \cdot \xi$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Quotient construction

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Define $\xi \sim_\mu \zeta$ if

$$d_{\mathcal{W}}\left((\pi_x + h\pi_v)\#\xi, (\pi_x + h\pi_v)\#\zeta\right) = o(h).$$

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \overset{\text{isometry}}{=} \mathscr{P}_2(\mathbb{R}^d)_\mu / \sim_\mu .$$

Tangent cone: quotient approach

Construction of the tangent cone

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Consider

- the $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that

$$s \mapsto (\pi_x + s\pi_v)\#\xi$$

  is a geodesic;
- the positive cone $\alpha \cdot \xi$ for all $\alpha \geqslant 0$,
- the completion of the previous cone with respect to W_μ.

The resulting set is denoted $\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$.

Quotient construction

Let $\mu \in \mathscr{P}_2(\mathbb{R}^d)$. Define $\xi \sim_\mu \zeta$ if

$$d_{\mathcal{W}}\left((\pi_x + h\pi_v)\#\xi, (\pi_x + h\pi_v)\#\zeta\right) = o(h).$$

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \overset{\text{isometry}}{=} \mathscr{P}_2(\mathbb{R}^d)_\mu / \sim_\mu .$$

That's it!

Thank you!

[AGS05] Luigi Ambrosio, Nicola Gigli, and Guiseppe Savaré.
Gradient Flows.
Lectures in Mathematics ETH Zürich. Birkhäuser-Verlag, Basel, 2005.

[Bre91] Yann Brenier.
Polar factorization and monotone rearrangement of vector-valued functions.
Communications on Pure and Applied Mathematics, 44(4):375–417, June 1991.

[Gig08] Nicola Gigli.
On the Geometry of the Space of Probability Measures Endowed with the Quadratic Optimal Transport Distance.
PhD thesis, Scuola Normale Superiore di Pisa, Pisa, 2008.

[Lad87] Ol'ga A. Ladyženskaja.
The Mathematical Theory of Viscous Incompressible Flow.
Number 2 in Mathematics and Its Applications. Gordon and Breach, 1987.