A geometric question arising from Hamilton-Jacobi-Bellman equations in $\mathscr{P}_2$

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Università di Pisa

Table of Contents

Classical HJB

We are interested in solutions v of equations of the form

$$-\partial_t v(t,x) + \sup_{u\in U} - \langle \nabla_x v(t,x), f(x,u)\rangle = 0, \qquad v(T,\cdot) = \mathfrak{J}. \tag{1}$$

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A main result of HJB theory is that under suitable regularity, V is the unique viscosity solution of (1).

Viscosity solutions

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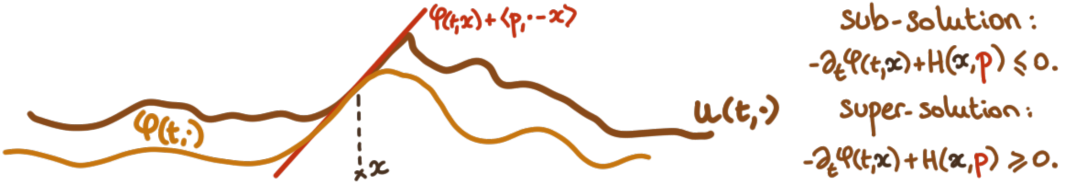

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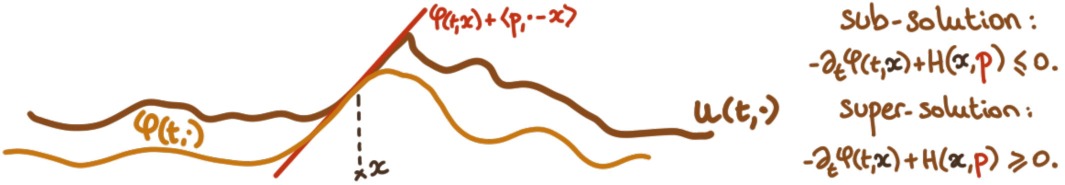

What about spaces without smooth functions?

Table of Contents

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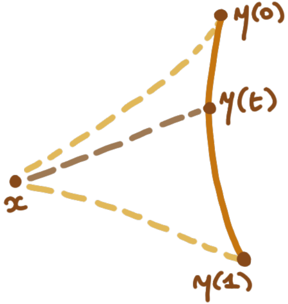

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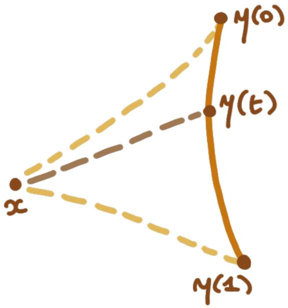

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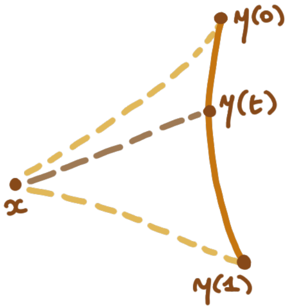

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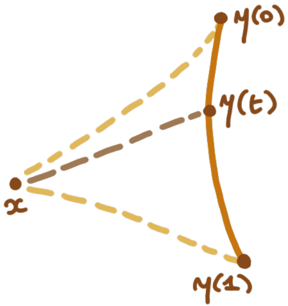

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- Examples: $\mathbb{R}^d$, 1-d network without loops, Euclidean gluings. HJB equations there (Achdou, Camilli, Cutrì, Tchou, Schieborn, Marchi, Imbert, Monneau, Zidani, Barles, Briani, Chasseigne, Girard, Forcadel, Cardaliguaguet...)

Canonical construction of tangent cones

Manipulating the squared distance inequality, we get that for two geodesics γ, γ' issued from the same x,

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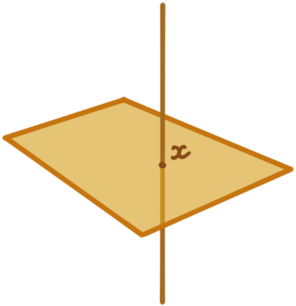

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- Provides curves along which one can differentiate.

HJ equations in CAT(0) spaces

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$$\boxed{\text{Take test functions that are } \begin{cases}\text{semiconvex}\\ \text{semiconcave}\end{cases} \text{ when touching from } \begin{cases}\text{above}\\ \text{below}\end{cases}.}$$

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Theorem – (Jerhaoui & Zidani 2023) Comparison principle, stability, Perron's method.

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Based on mutations of [Aub99][1]. Using the latter results and [FL23][2], **one gets well-posedness of the control system** with satisfactory estimates, par to Cauchy-Lipschitz.

[1] J. P. Aubin, *Mutational and Morphological Analysis* (1999).

[2] H. Frankowska and T. Lorenz, "Filippov's Theorem for mutational inclusions in a metric space" (2023).

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- *By definition*, each curve $(y_t)_t$ is well-approximated by a reparameterized geodesic.

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Table of Contents

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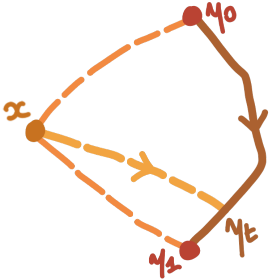

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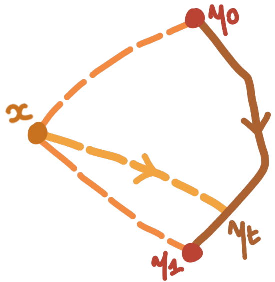

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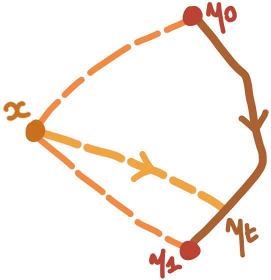

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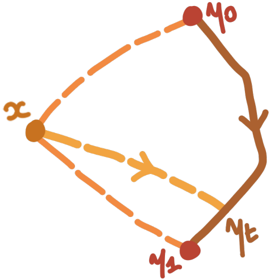

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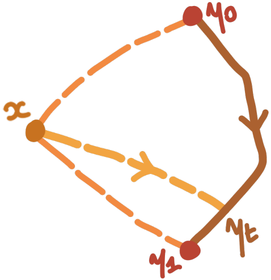

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Results on viscosity solutions

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Theorem – Comparison principle [Aus25][1] With H as above, any semi-continuous locally bounded subsolution u and supersolution v such that $u(T, \cdot) \leqslant v(T, \cdot)$ satisfy $u \leqslant v$.

Quid of control problems in (some) CBB case?

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Continuity equations

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General feeling: no hope for stability if the mass can split.

Linearization by straight line

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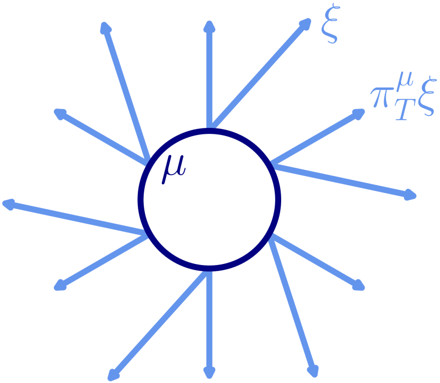

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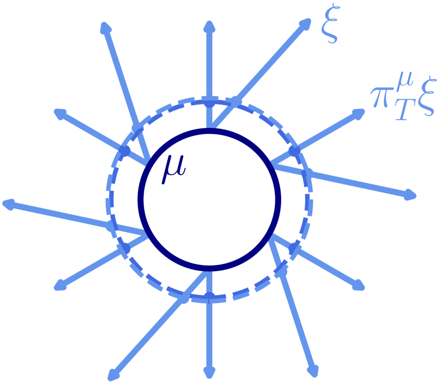

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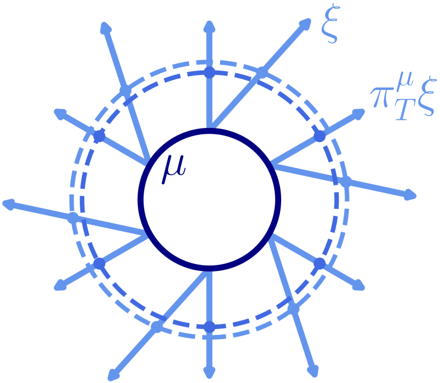

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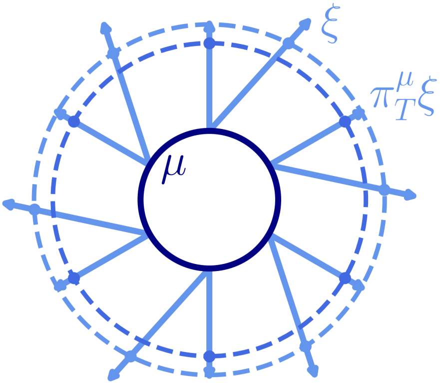

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Corollary The value function $V(t,\nu) := \inf_{u(\cdot)\in L^0([t,T];U)} \mathfrak{J}(\mu_T^{t,\nu,u})$ is a viscosity solution of

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Supersolution: similar, replacing regularity by an approximation along a subsequence (hence conv).

Table of Contents

What is not tangent

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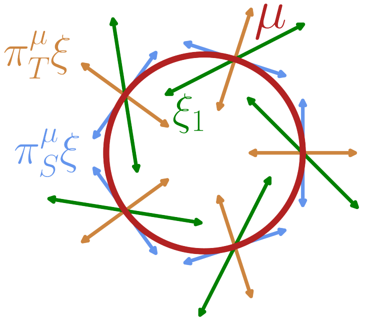

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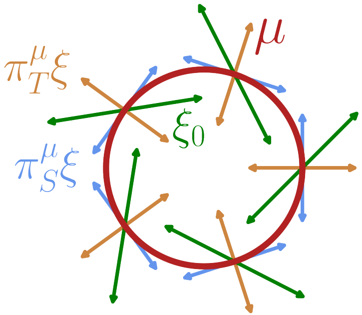

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Worst case possible!

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- Avoidance theorems saying that a continuity equation starting outside of this set remains outside?

Thank you for your attention.