The simple beauty of the eikonal equation

Humble introduction to Lax-Oleĭnik analysis for some Hamilton-Jacobi equations

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LMRS doctoral seminar

Table of Contents

Definitions

The eikonal equation Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Find $u \in W^{1,\infty}(\Omega)$ s.t.

$$\begin{cases} \|\nabla u(x)\| = 1 & x \in \Omega, \\ u(x) = u_b(x) & x \in \partial\Omega. \end{cases}$$

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(Our) Hamilton-Jacobi equation Let $H : \mathbb{R}^n \mapsto \mathbb{R}$ be convex, lsc and proper, $\Omega \subset \mathbb{R}^n$ be open with suitable boundaries, and $n \in W^{1,\infty}(\Omega, [\inf H, \overline{n}])$. Find $u \in W^{1,\infty}(\Omega)$ s.t.

$$\begin{cases} H(\nabla u(x)) = n(x) & x \in \Omega, \\ u(x) = u_b(x) & x \in \partial\Omega. \end{cases}$$

What is a solution?

Toy example in dimension $n = 2$, with

$$H(x,y) := x + |y|, \quad \Omega :=]0,\infty[\times \mathbb{R}, \quad \partial\Omega = \{0\} \times \mathbb{R}, \quad n \equiv 0, \quad u_b(x,y) := e^{-y^2}.$$

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$$\begin{cases} \partial_x u(x,y) + |\partial_y u(x,y)| = 0 & x > 0,\ y \in \mathbb{R} \\ u(x,y) := e^{-y^2} & x \in \{0\},\ y \in \mathbb{R}. \end{cases}$$

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If pointwise sense, we have

- $|\partial_y u_b(0, 0)| = 0$, so $\partial_x u(x, y) = 0$, and $u(x, 0) = u_b(0, 0) = 1$ for all $x \geqslant 0$.

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- $|\partial_y u_b(0,0)| = 0$, so $\partial_x u(x,y) = 0$, and $u(x,0) = u_b(0,0) = 1$ for all $x \geqslant 0$.
- $|\partial_y u_b(0,y)| > 0$ if $y \neq 0$, so $\partial_x u(x,y) < 0$, and the solution is discontinuous at $y = 0$.

What is a solution

Viscosity solution A function u is a subsolution (resp. supersolution) if $\forall x \in \Omega$, $\forall p \in \partial^+ u(x)$ (resp. $\forall p \in \partial^- u(x)$), we have

$$\begin{cases} H(p) \leqslant n(x) \quad H(p) \geqslant n(x) & x \in \Omega, \\ u(x) \leqslant u_b(x) \quad u(x) \geqslant u_b(x) & x \in \partial\Omega. \end{cases}$$

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On our example:

- $u \equiv 0$ is subsolution (exactly satisfies the equation, and $u \leqslant u_b$ on the boundary).
- $u(x, y) := u_b(x, y)$ is supersolution ($\partial_x u = 0$, $|\partial_y u| \geqslant 0$ and exact boundary condition).

Viscosity solutions are maximal

Proposition – [Lio82] Viscosity solutions may be build as the pointwise maximum of all Lipschitz-continuous viscosity subsolutions.

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- Precisely the construction used in Perron's method (see for instance [For22]).
- Hints an underlying convention (why not min of subsolutions?).
- Historically, rooted in vanishing viscosity method (see [Eva10]), with the sign convention

$$H(\nabla u(x)) - \varepsilon \Delta u(x) = 0.$$

Table of Contents

Idea on the eikonal equation (1/2)

Suppose that u is a lipschitz subsolution. For all $T > 0$,

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We have $\forall q \in \mathbb{R}^n$ that $\max_{p \in \mathbb{R}^n} \left[q \cdot p - \delta_{\{|p| \leqslant 1\}} \right] = \max_{|p| \leqslant 1} q \cdot p = |q|.$

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Idea on the eikonal equation (2/2)

Candidate solution We define

$$u(x) := \inf_{y \in \partial\Omega, T>0} \left[u_b(y) + T \left(1 + \delta_{\left\{ \frac{|y-x|}{T} \leqslant 1 \right\}} \right) \right] .$$

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Theorem The candidate solution is indeed the viscosity solution.

Legendre transform

Let $H : \mathbb{R}^n \mapsto \mathbb{R}$ be convex, l.s.c and proper. Then $H^*(q) := \sup_{p\in\mathbb{R}^n} [\langle q, p\rangle - H(p)]$.

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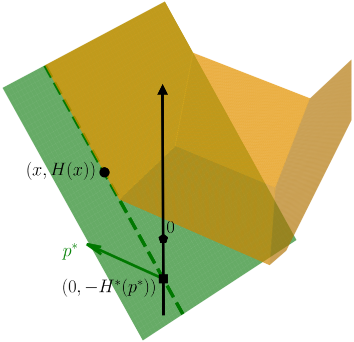

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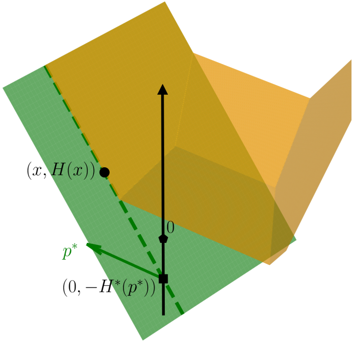

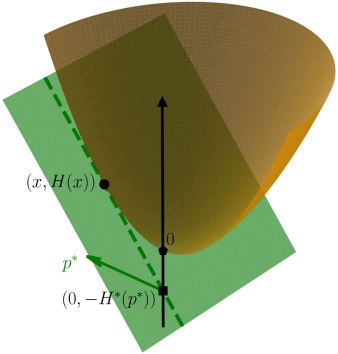

General representation theorem

Candidate solution for the HJ equation We define $u(x) := \inf_{y \in \partial\Omega} \left[u_b(y) + \mathcal{L}(y,x) \right]$,

$$\mathcal{L}(y,x) := \inf_{\substack{T>0,\ \gamma \in W^{1,\infty}([0,T],\Omega) \\ \gamma(0)=y,\ \gamma(T)=x}} \left[\int_{s=0}^{T} \left[n(\gamma(s)) + H^*(\dot{\gamma}(s)) \right] ds \right].$$

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Theorem – Oleĭnik-Hopf [Ole63, Hop65] The candidate u is the viscosity solution.

Table of Contents

The toy problem (1/2)

Step 1 Compute the convex dual of $H(x,y) = x + |y|$.

$$H^*(\alpha,\beta) = \max_{x,y\in\mathbb{R}^2} x\alpha + y\beta - (x + |y|) = \delta_{\{\alpha=1\}} + \delta_{\{|\beta|\leqslant 1\}}.$$

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$$\mathcal{L}((0, z), (x, y)) = \inf_{T>0, \gamma:(0,z)\to(x,y)} \int_0^T 0 + H^*(\dot\gamma(s))ds = \begin{cases} 0 & \text{if } \dot\gamma \in \{1\} \times [-1, 1], \\ \infty & \text{otherwise.} \end{cases}$$

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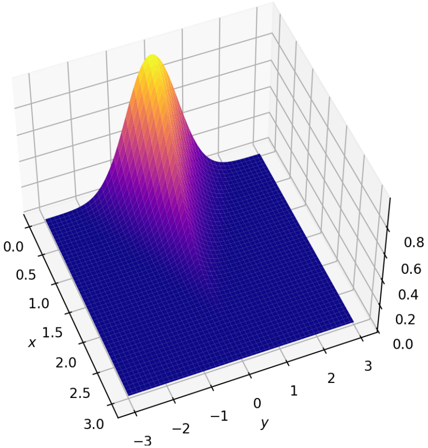
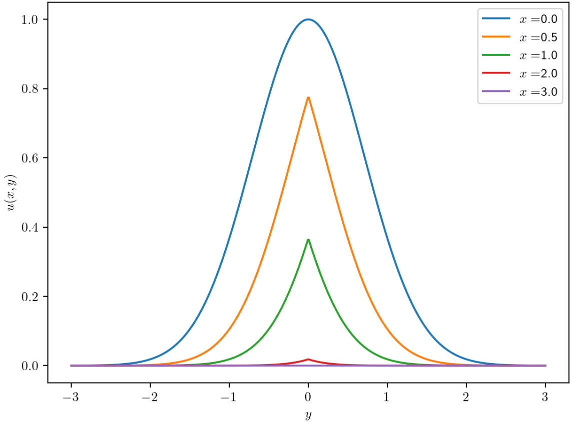
Step 3 Apply Lax-Oleĭnik formula:

$$u(x,y) = \min_{(0,z)\in\{0\}\times\mathbb{R}} \left[u_b(0,z) + \mathcal{L}((0,z),(x,y))\right] = \min_{z\in y+x[-1,1]} e^{-z^2} = e^{-(|y|+x)^2}.$$

The toy problem (2/2)





Some feeling-good pictures

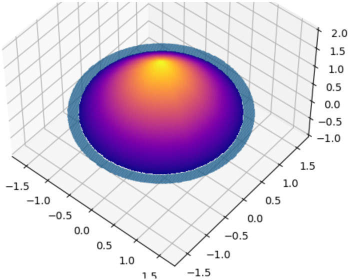


Figure: Euclidian distance to {circle, , } with {null, , } boundary condition.

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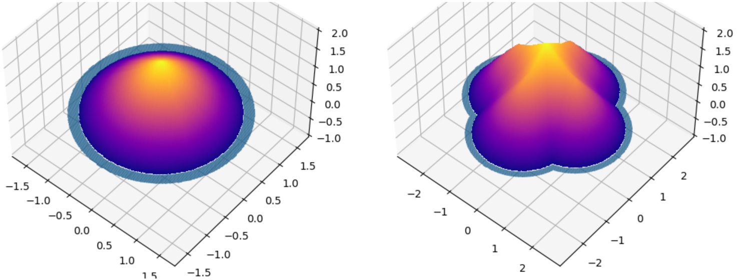

Figure: Euclidian distance to {circle, trefle, } with {null, null, } boundary condition.

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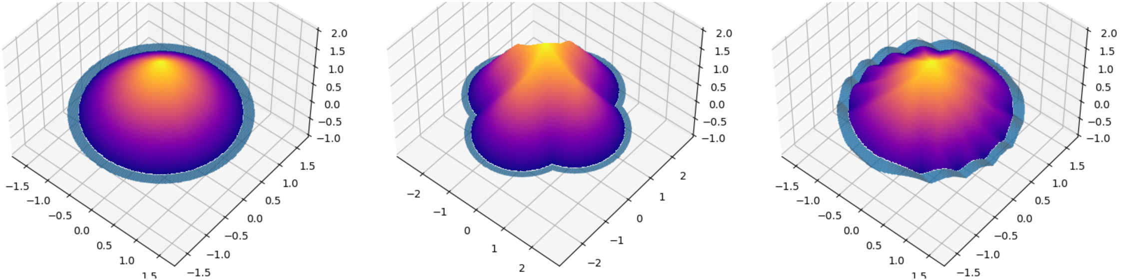

Figure: Euclidian distance to {circle, trefle, circle} with {null, null, funny} boundary condition.

Critical boundary conditions for the eikonal equation

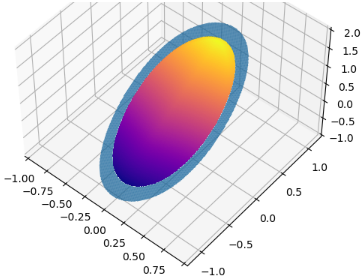

Figure: Boundary conditions that are {1-lipschitz, , }.

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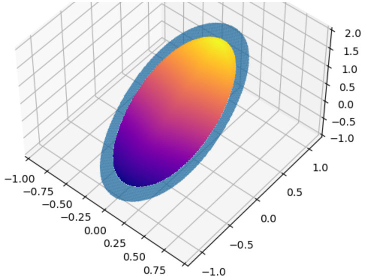

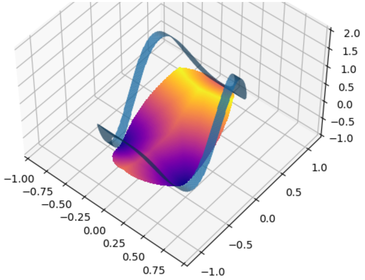

Figure: Boundary conditions that are {1-lipschitz, not 1-lipschitz, }.

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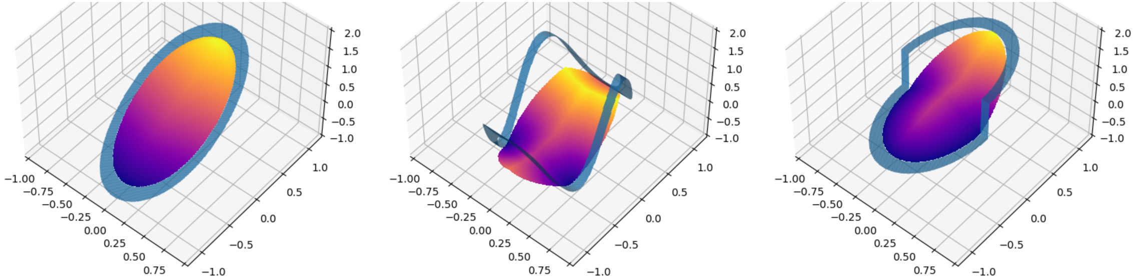

Figure: Boundary conditions that are {1-lipschitz, not 1-lipschitz, discontinuous}.

Thank you!

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