Second blow-ups of squared Wasserstein distances in dimension one

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Università di Pisa

The problem

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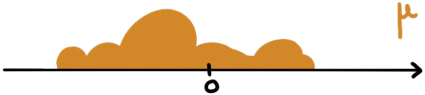

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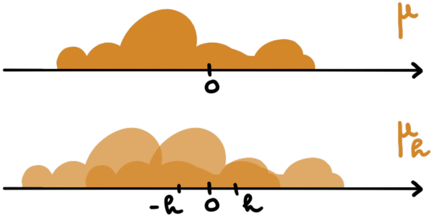
Consider the mean μ_h of two copies of μ, respectively shifted by $\pm h$. Explicitly,

$$\mu_h = \frac{1}{2}\left[(id - h)_\# \mu + (id + h)_\# \mu\right].$$



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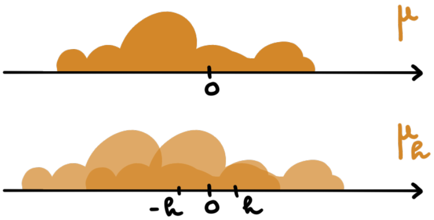

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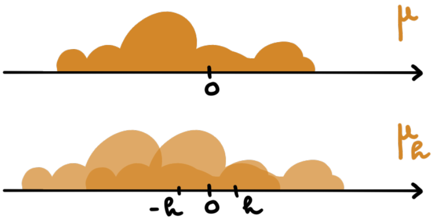
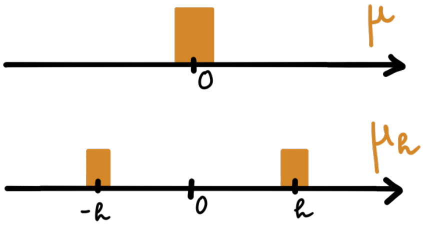


Question When does it hold that

$$\limsup_{h \searrow 0} \frac{W^2(\mu, \mu_h)}{h^2} = 1 \qquad ?$$

Examples

- **Example 1** Consider $\mu = \delta_0$. Then $\mu_h = \frac{1}{2}\left[\delta_{-h} + \delta_h\right]$.



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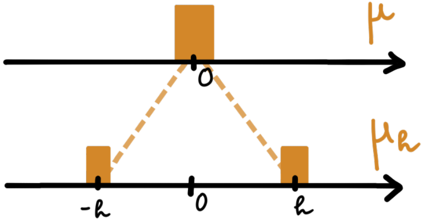

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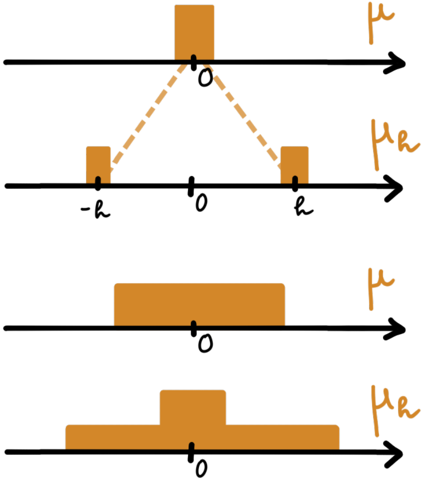

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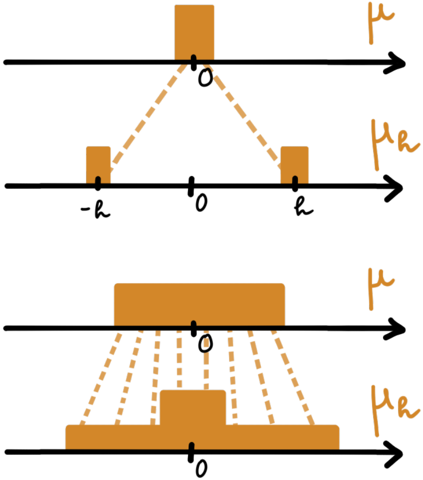

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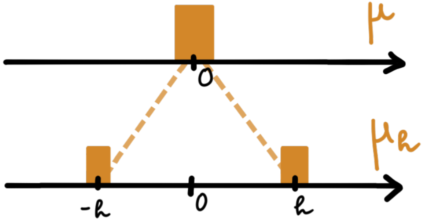

• **Example 2** Consider $\mu = \mathcal{L}_{[-1/2, 1/2]}$.
After computation, one has $W^2(\mu, \mu_h) = ch^3$, so that

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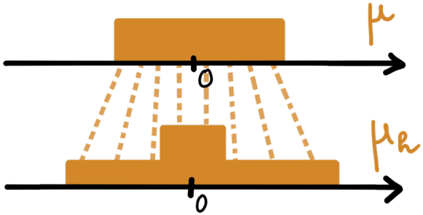

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☛ In the question, we expect μ to be "concentrated".

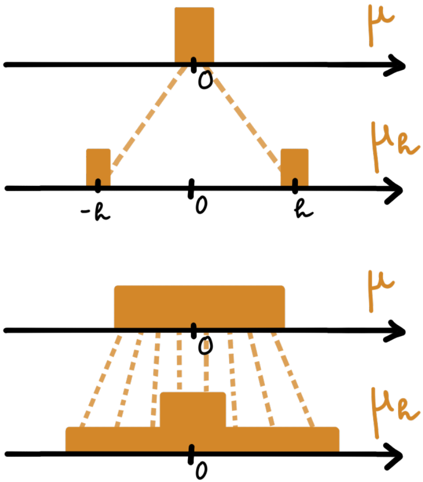

Table of Contents

Original: PDEs in the Wasserstein space

Studying PDE formulations of optimal control problems:

$$-\partial_t u(t,\mu) + H\left(\mu, D_\mu u(t,\mu)\right) = 0, \qquad + \text{ terminal condition.}$$

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Vast literature, with different techniques. Very non-exhaustive list for first-order case only:

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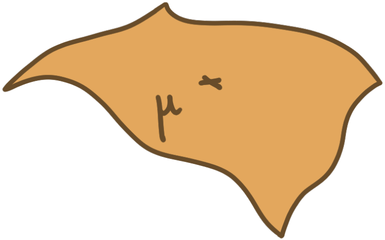
- **Metric tools:** Ambrosio & Feng 2014, Gangbo & Święch (2014, 2015), Giga, Hamamuki, Nakayasu 2015, Conforti, Kraaij & Tonon (2023, 2023, 2024)

Taking directional derivatives

- General theory in Alexandrov spaces: $D_\mu u(\xi)$ is well-defined if ξ lies in a suitable *tangent cone* $\mathbf{Tan}_\mu$.



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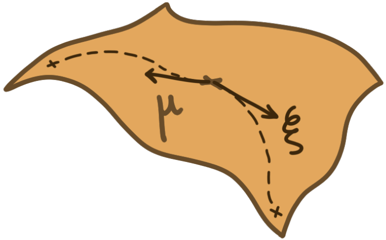

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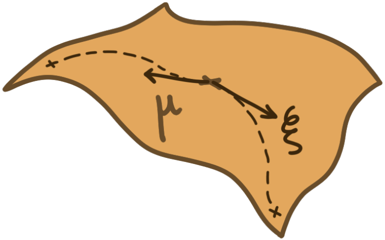
Then usual definition

$$D_\mu u(\xi) := \lim_{h \searrow 0} \frac{u(\mu \dotplus h\xi) - u(\mu)}{h},$$

where $h \mapsto \mu \dotplus h\xi$ is the curve with velocity ξ.



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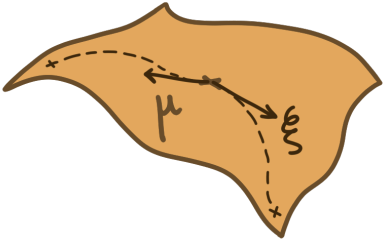

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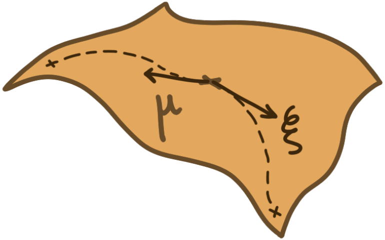

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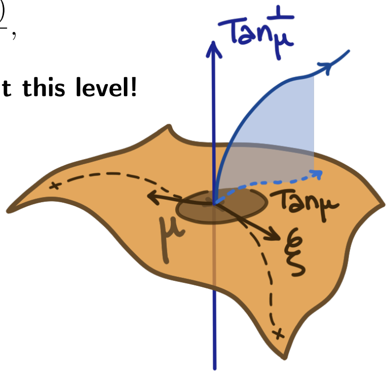
⚠ $\mathscr{P}_2(\mathrm{T}\,\mathbb{R}^d)_\mu$ is usually strictly larger than $\mathbf{Tan}_\mu$; we can even define an "orthogonal" $\mathbf{Tan}_\mu^\perp \subset \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$.



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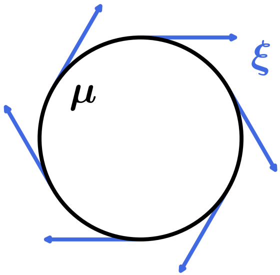

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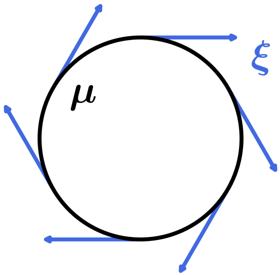

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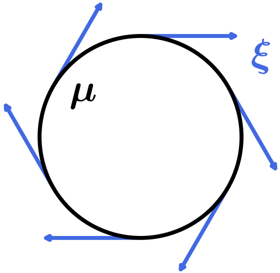

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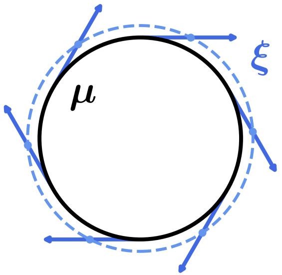

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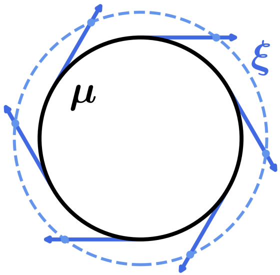

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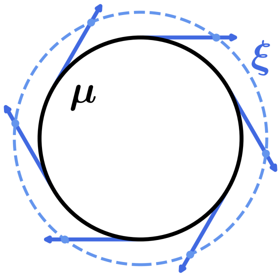

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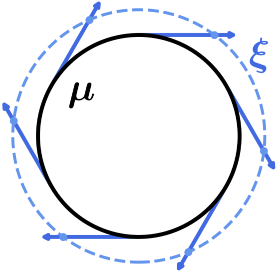

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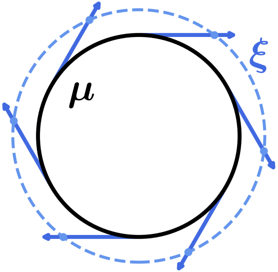

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WRONG

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Partial positive result

Theorem If ξ is deterministic, i.e. of the form $(id, g)_{\#}\mu$ for some vector field $g \in L^2_\mu$, then

$$\xi \in \mathbf{Tan}_\mu^\perp \iff -\operatorname{div}(g\mu) = 0 \text{ distributionally} \iff \lim_{h \searrow 0} \frac{W^2(\mu, \mu \nrightarrow h\xi)}{h^2} = 0.$$

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☛ Saves the strategy for the PDE problem in the case under consideration. Other choices: use $\mathcal{C}^1$ test functions, penalize (Daudin & Seeger 2025), bypass the geometric tangent cone (Bertucci 2025)...

General negative result

There exists $\mu \in \mathscr{P}([0,1])$ s.t. the unit symmetric measure field $\xi := \frac{1}{2}\left[(id,-1)_{\#}\mu + (id,1)_{\#}\mu\right]$ belongs to $\mathbf{Tan}_{\mu}^{\perp}$, and

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Definition A measurable set $A \subset \mathbb{R}$ belongs to the class $\mathcal{A}$ if there exist sequences $(s_n)_n \subset (0,1)$ and $(\tau_n)_n \subset [0,1)$ going to 0 such that for any $n \in \mathbb{N}$, and any $x \in A$,

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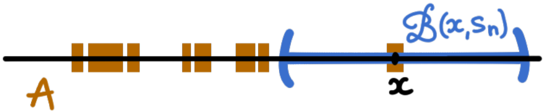

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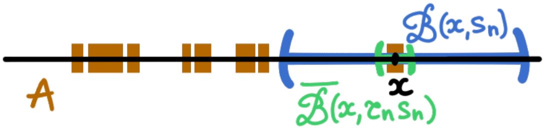

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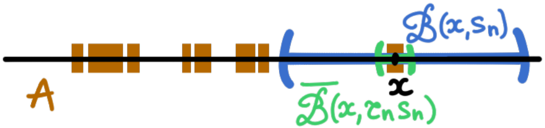

For instance: finite sets, some (uncountable) skinny Cantor sets.

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Key argument: for any $\varphi : \mathbb{R} \to \mathbb{R}$ and $h > 0$, let

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So $|y-x|$ is either smaller than ch, or larger than Ch, for c, C depending only on γ.

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Now classical trick of taking intersections of sets of mass larger than $1-\varepsilon 2^{-(k+1)}$ and recover a set of mass larger than $1-\varepsilon$. □

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- *Does not say* whether $\xi \in \mathbf{Tan}_\mu^\perp$ means 0 initial velocity...

Open problem

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If $\mu \ll \mathcal{L}$, the property holds, but this is not sharp.

Thank you for your attention

Talk based on the ArXiv preprint "Characterization of measures on the real line that are critically unstable under small shifts" (2026).