Optimal control problems and Hamilton-Jacobi-Bellman equations in some curved metric spaces

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supervised by
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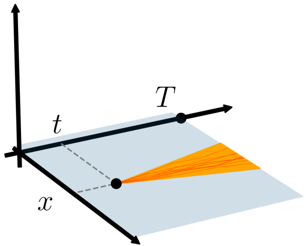

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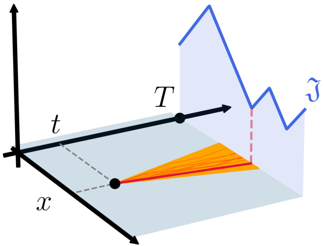

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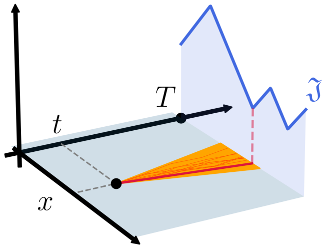

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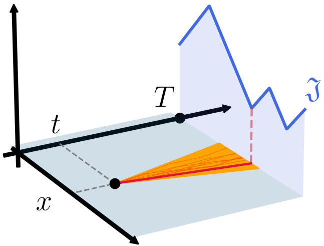

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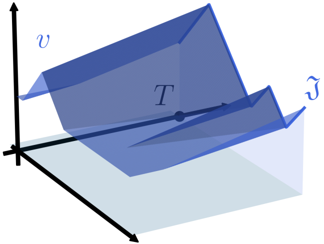

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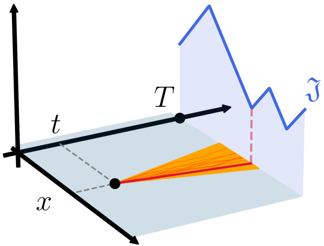

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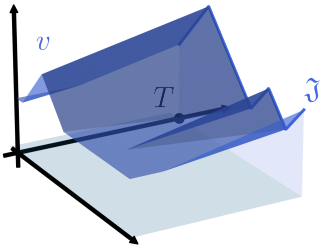

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Control of measures [CQ08, GNT08, HK15, CMNP18, JMQ20, JMQ23, JJZ24] in Wasserstein spaces.

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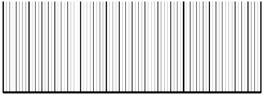

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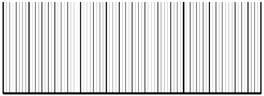

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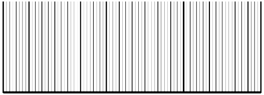

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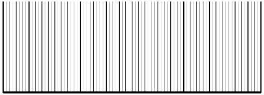

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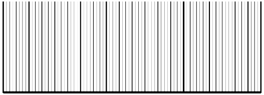

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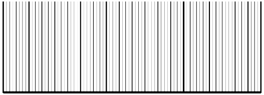

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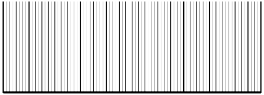

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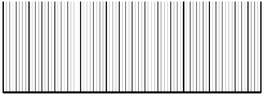

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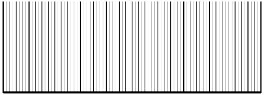

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Combining properties of gradient flows and mutations, well-posedness.

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Usual argument:

- consider a sequence $(y_n, \dot{y}_n)_n$ of solutions,
- extract of a strong-weak limit (y, b),
- show that $y_t = x + \int_0^t b_s ds$ and $b_s \in \overline{\mathrm{conv}} f(y_s)$.

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Proposition For such f, y solves $\mathring{y}_t \cap f(y_t) \neq \emptyset$ iff some measurable selection $(E_t)_t$ of $t \mapsto f(y_t)$ satisfies

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This definition supports a strong comparison principle (Theorem 3.1.12, [AJZ24]).

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$\left(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}}\right)$ is CBB(0), so we can turn to the Mayer problem

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Theorem Assume $f, \mathfrak{J}$ Lipschitz. Then the value function V is the unique viscosity solution of

$$\begin{cases} -\partial_t v(t,\mu) + H\left(\mu, D_\mu v(t,\mu)\right) = 0 \\ v(T,\cdot) = \mathfrak{J} \end{cases}$$

where $H(\mu, p) := \sup_{b \in \overline{\operatorname{conv}} f[\mu, U]} -p(\pi_T^\mu b)$.

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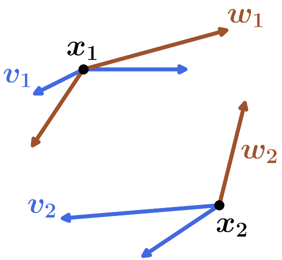

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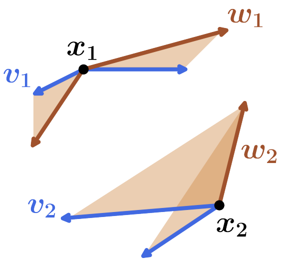

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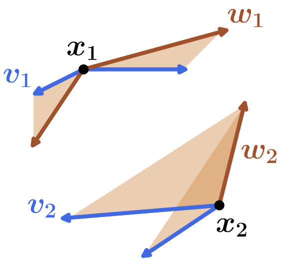

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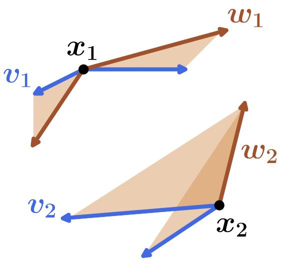

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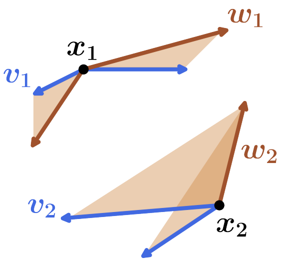

Def The tangent cone $\mathbf{Tan}_\mu$ is

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Classification

Denote by $h \mapsto \exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)_\# \xi$ the exponential of $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$.

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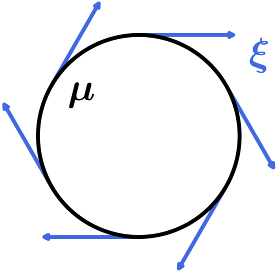

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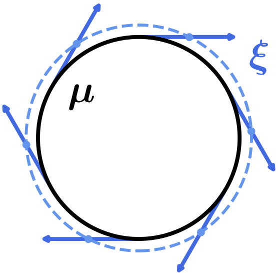

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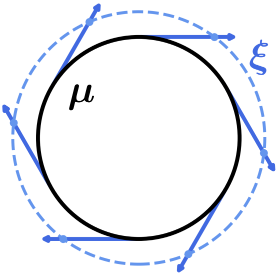

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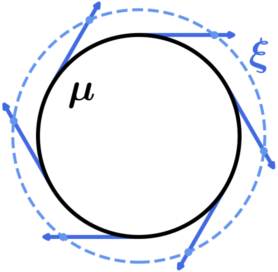

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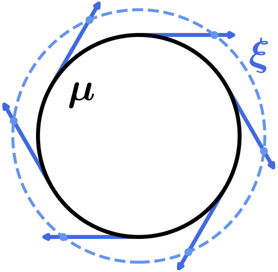

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Further precisions in dimension one:

- $(\mathrm{E_T})$ and $(\mathrm{E_S})$ hold if μ is purely atomic or absolutely continuous.

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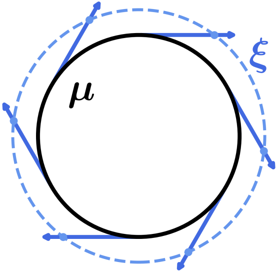

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Yields $d_{\mathcal{W}}\big(\exp_\mu(h\cdot f_\#\mu), \exp_\mu(h\cdot\pi^\mu_T f_\#\mu)\big) = o(h)$, hence the post-it lemma.

Further precisions in dimension one:

- $(\mathrm{E_T})$ and $(\mathrm{E_S})$ hold if μ is purely atomic or absolutely continuous.
- Neither $(\mathrm{E_T})$ or $(\mathrm{E_S})$ holds in general.

Decomposition of a measure

Remark If $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ are centred, $\langle \xi, \zeta \rangle_\mu = 0$ if and only if $\langle \xi_x, \zeta_x \rangle_{\delta_x} = 0$ μ−a.e..

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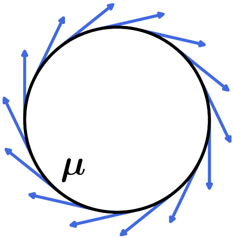

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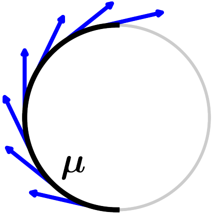

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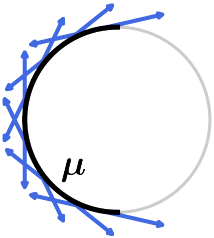

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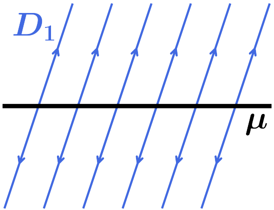

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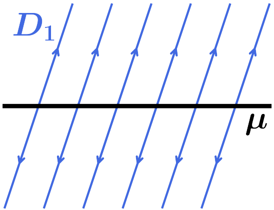

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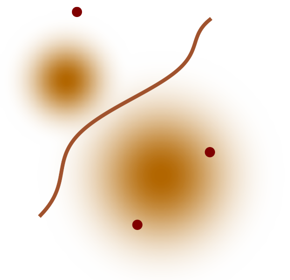

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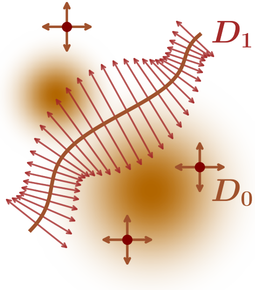

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For instance, $\varphi := d^2_{\mathcal{W}}\left(\cdot, \frac{\delta_{-1}+\delta_1}{2}\right)$ has

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Conjecture: the decomposition $\mu = \sum_{k=0}^{d} m_k \mu^k$ satisfies

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Hence μ^0 would be the atoms, μ^1 concentrated on c − c curves, etc...

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Other tangent cones

Similar yet distinct decompositions and spaces exist in relation with Lipschitz functions [BCJ05, AM16]. Any link?

Thank you!

Details on the HJB formulation in CAT(0) spaces

Let $H : (x,p) \mapsto \sup_{\varphi \in \overline{\mathrm{conv}} f(x)} -p(\nabla_x \varphi)$, and consider the following Hamilton-Jacobi-Bellman equation:

$$\text{(HJB)} \qquad -\partial_t v + H(x, D_x v(t,x)) = 0, \qquad v(T,\cdot) = \mathfrak{J}.$$

Def [JZ23] A viscosity solution $v \in \mathcal{C}([0,T] \times \mathbb{R}^d; \mathbb{R})$ of (HJB) is both a
subsolution: if φ is $\mathcal{C}^2$ in time, **semiconvex** in space, and touches v from **above** at (t,x),

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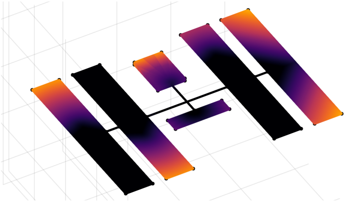
Under the technical assumption **[A2.1.3]** to approximate the gradient flows of functions in $\mathcal{E}$ by geodesics, the following holds.

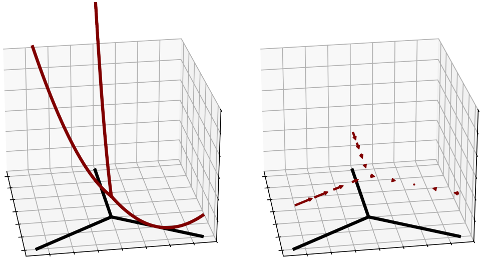
Proposition Assume $f, \mathfrak{J}$ Lipschitz and bounded. Then V is the unique viscosity solution of (HJB).



Details on the convexification procedure

A dynamic f is valued in a set $\mathcal{E} \subset \mathrm{Lip}(\mathbb{R}^d;\mathbb{R})$ of concave functions. Its closed convex hull is defined in the Banach space $\mathbb{E}$ of limit points of Lipschitz DC functions, quotiented by constants, with respect to

$$\|\varphi\|_{1,\infty} := \sup_{(x,v)\in \mathrm{T}\mathbb{R}^d, |v|_x=1} \left|D_x\varphi(v)\right|.$$



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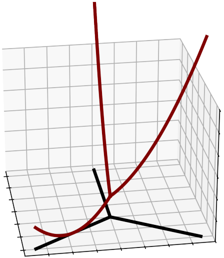

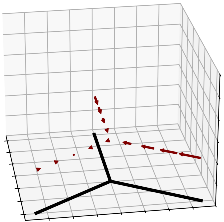

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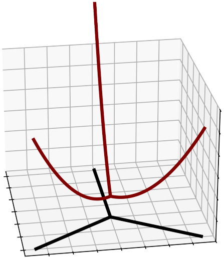

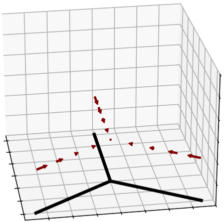

Proposition Assume that each $\varphi \in \mathcal{E}$ satisfies $D_x\varphi = \langle \nabla_x\varphi, \cdot\rangle_x$. If $(y_t)_t$ solves $\mathring{y}_t \ni \int_{\varphi\in\mathcal{E}} \varphi \, d\omega_s(\varphi)$ for $\omega \in L^1(0,T;\mathscr{P}_1(\mathcal{E}))$, then

$$y_t^+ = \mathrm{Bary}_{\mathrm{T}_{y_t}\mathbb{R}^d}\left(\nabla_{y_t\#}\omega_t\right) \quad a.e. \text{ in } [0,T].$$

Details on the comparison principle in CBB(0) spaces

We consider

$$-\partial_t v(t,x) + H(x, D_x v(t,x)) = 0, \qquad v(T,\cdot) = \mathfrak{J}.$$

The notion of viscosity solution is based on semiconcave/semiconvex test functions.

Theorem Assume $H(y, -\lambda D_y d^2(x,\cdot)) - H(x, \lambda D_x d^2(\cdot, y)) \leqslant \lambda C d(x,y)(1+d(x,y))$ for $\lambda \geqslant 0$, and $H(x,\cdot)$ Lipschitz. Let $u, -v$ be locally uniformly upper semicontinuous and locally bounded, with u subsolution and v supersolution. Then $\sup_{[0,T]\times\Omega} u - v \leqslant \sup_\Omega u(T,\cdot) - v(T,\cdot)$.

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Here

- "strong" upper semicontinuity is equivalent to $B \mapsto \sup_B u$ upper semicontinuous in the Hausdorff topology over nonempty compact sets.
- The argument employs the Ekeland-Borwein-Preiss-Zhu principle [BZ05].
- Growth conditions are avoided owing to the variable t and a clever penalization from [FGŚ17].

Details about the classification $\mathbf{Tan}_\mu/\mathbf{Sol}_\mu$

The equivalences

$$\xi \in \mathbf{Tan}_\mu \Leftrightarrow \lim_{h\searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h\cdot\xi))}{h} = \|\xi\|_\mu \qquad \text{and} \qquad \zeta \in \mathbf{Sol}_\mu \Leftrightarrow \lim_{h\searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h\cdot\zeta))}{h} = 0$$

hold

- if ξ, ζ are induced by maps;
- in dim 1, if μ is purely atomic or absolutely continuous with respect to the Lebesgue measure.

All results in this directions are consequences of the following lemma:

Let $\xi \in \mathscr{P}_2(\mathrm{T}\Omega)_\mu$ such that $\lim_{h\searrow 0} \frac{d_{\mathcal{W}}(\mu, \exp_\mu(h\cdot\xi))}{h} = \|\xi\|_\mu$. Then there exists $(h_n)_n \searrow 0$ such that

$$\lim_{n\to\infty} \sup_{\gamma \in \frac{1}{h_n}\cdot \exp_\mu^{-1}(\exp_\mu(h_n\cdot\xi))} d_{\mathcal{W},\mathrm{T}\Omega}\left(\gamma, \xi\right) = 0.$$

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Decompose $\mu = m_a \mu^a + m_d \mu^d$, with $\mu^a \in \mathscr{P}_2(\mathbb{R})$ purely atomic and $\mu^d \in \mathscr{P}_2(\mathbb{R})$ diffuse (atomless).

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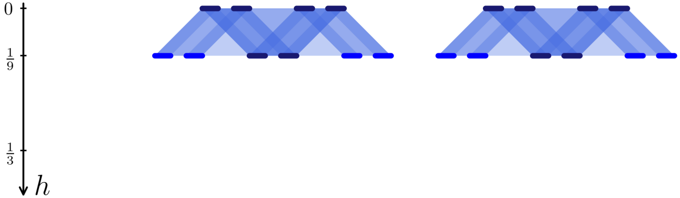

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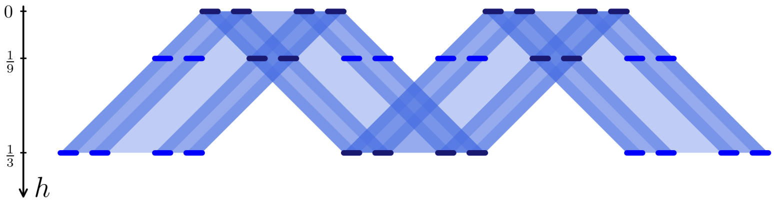

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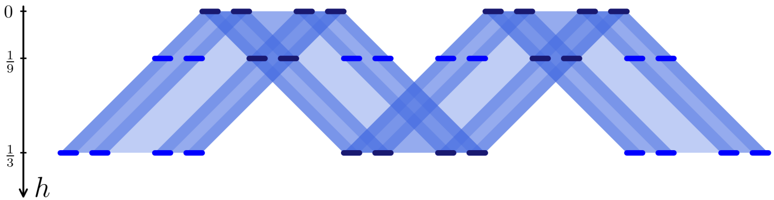

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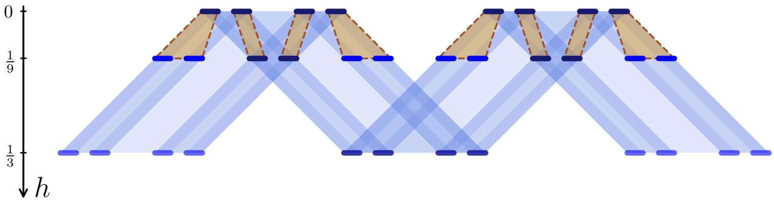

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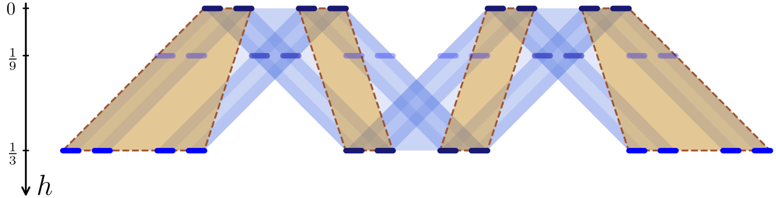

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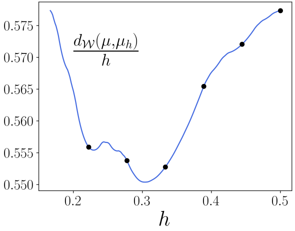
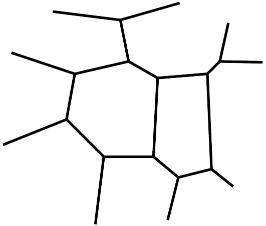


Details on the directional differentiability in CAT(0) spaces (1/3)

If $\mathbb{R}^d$ is a one-dimensional network, with possibly junctions and loops, $(\mathscr{P}_2(\mathbb{R}^d), d_{\mathcal{W}})$ does not have curvature bounds.



Question. Can we find directionally differentiable test functions?

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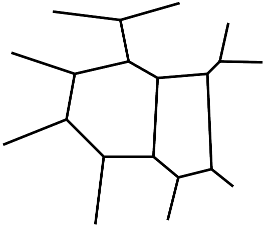

Question. Can we find directionally differentiable test functions?

Let $\mathscr{G} \subset \mathrm{AC}([0,1];\mathbb{R}^d)$ be the set of unit-speed geodesics, and $e_h : \mathscr{G} \to \mathbb{R}^d$ the evaluation at time $h \in [0,1]$, i.e. $e_h(\gamma) = \gamma_h$.

Theorem Let $\xi \in \mathscr{P}_2(\mathscr{G})$ and $\sigma \in \mathscr{P}_2(\mathbb{R}^d)$. Then

$$\lim_{h \searrow 0} \frac{d_{\mathcal{W}}^2((e_h)_\#\xi, \sigma) - d_{\mathcal{W}}^2((e_0)_\#\xi, \sigma)}{h} = \inf_{\substack{\alpha \in \Gamma(\xi,\sigma) \\ (e_0(\pi_\gamma), \pi_z)_\# \alpha \text{ opt.}}} \int_{(\gamma,z) \in \mathscr{G} \times \mathbb{R}^d} \frac{d}{dh}\Big|_{h=0} d^2(\gamma(\cdot), z)\, d\alpha(\gamma, z).$$

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General strategy: bound the limit sup from above (easy) and the limit inf from below (hard) by the same quantity.

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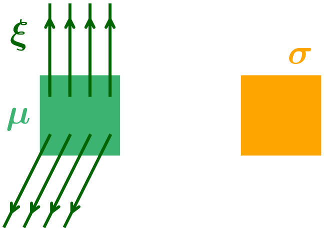

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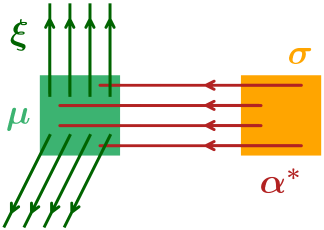

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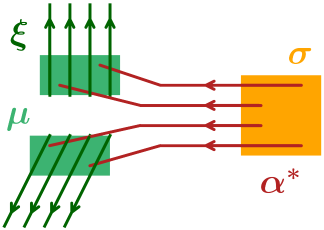

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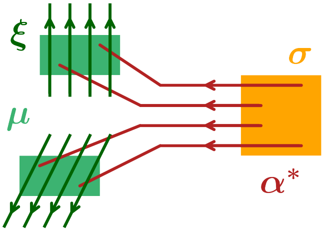

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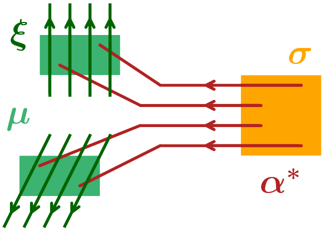

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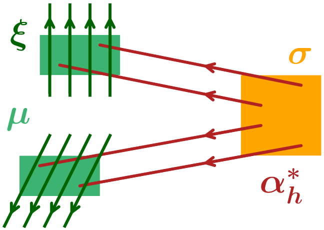

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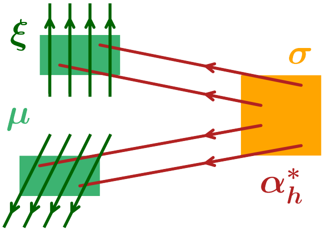

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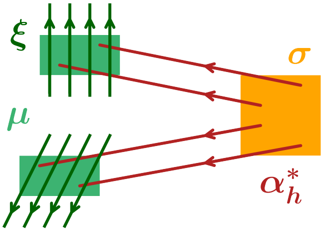

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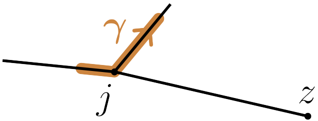

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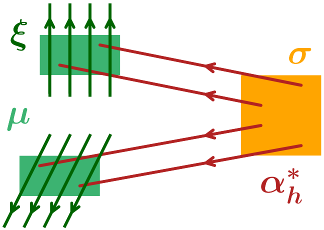

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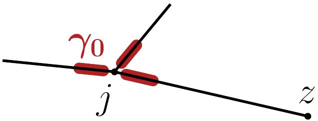

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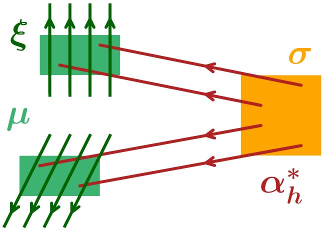

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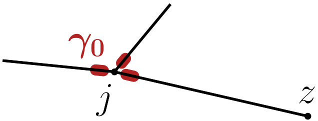

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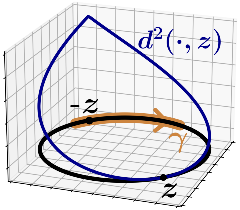

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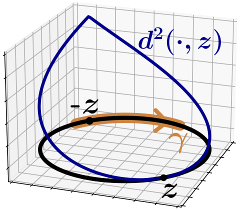

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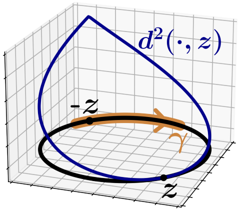
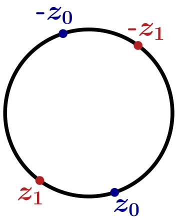
  - β is optimal *and* concentrated on pairs $(-z, z)$.





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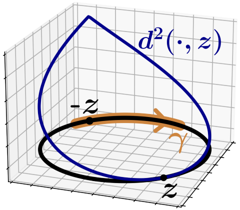

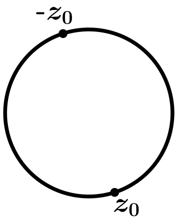

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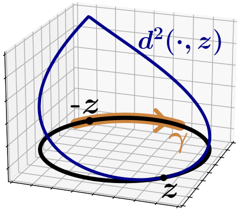

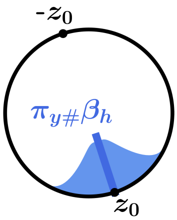

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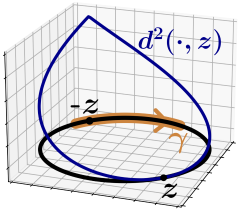

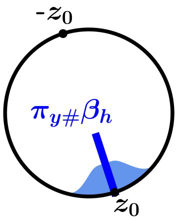

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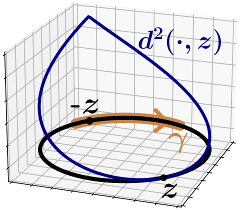

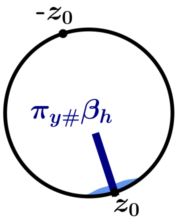

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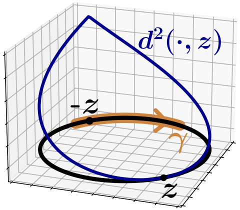

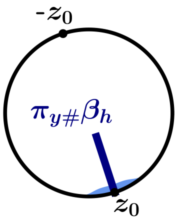

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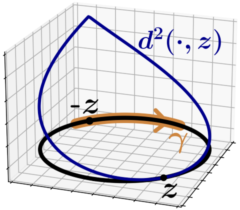

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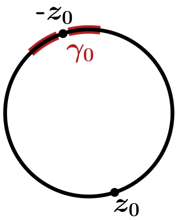

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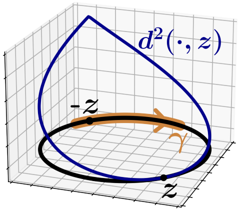

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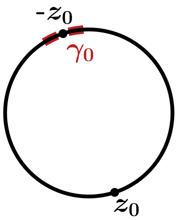

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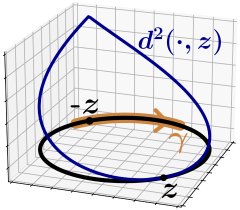

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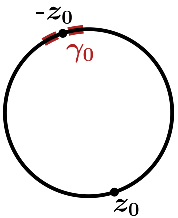

- **Conclusion.** The bad set vanishes, $d_{\mathcal{W}}^2(\cdot, \nu)$ directionally differentiable.

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