Decomposition of a measure according to Wasserstein tangent cones

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December 11, 2025
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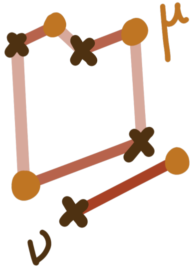

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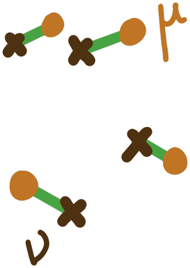
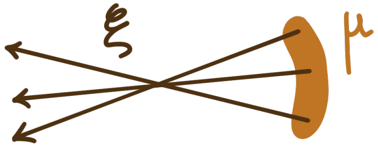


Perturbations of μ

One “moves around” $\mu \in \mathscr{P}_2(\Omega)$ along the curves

$$h \mapsto (\pi_x + h\pi_v)_\# \xi,$$

where $\xi = \xi(dx, dv) \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu$ satisfies $\pi_{x\#}\xi = \mu$.



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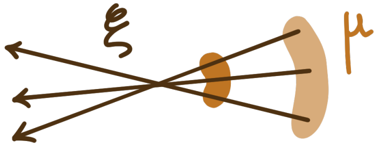

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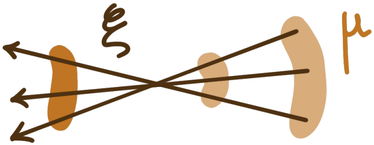

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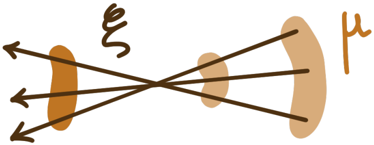

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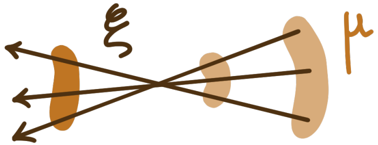

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$$\langle \xi_x, \zeta_x \rangle_x = \sup_{\eta_x \in \Gamma(\xi_x, \zeta_x)} \int_{(v,w)} \langle v, w \rangle \, d\eta_x.$$

Orthogonality and centred fields

Case of map-induced measure fields

If $\xi_i = (id, f_i)_{\#}\mu$ for $f_0, f_1 \in L^2_\mu(\Omega; \mathbb{R}^d)$, then

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What can we say on centred + "optimal" fields?

Examples

Definition – Centred tangent measure fields $\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega)^0_\mu$ is tangent if there exists optimal plans $(\eta_n)_n$ with first marginal μ, and $(h_n)_n \subset [1, \infty)$, such that $(\pi_x + h_n \pi_v)_\# \eta_n \to_n \xi$ with respect to W_μ. Denote $\mathbf{Tan}^0_\mu$ the set of such ξ.

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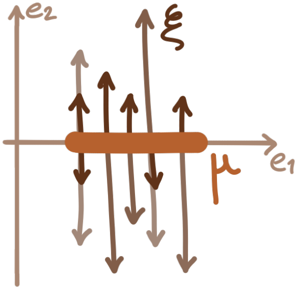
Ex. 3. If $\mu = (id, 0)_\# \mathcal{L}_{[0,1]}$ in dimension 2, then $\xi \in \mathbf{Tan}^0_\mu$ iff ξ is centred and $v \perp e_1$ for $\xi-$a.e. (x, v).



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(Extract of) Lott's result

Theorem 1.1 of [Lot16][1] If

- $\mathcal{M}$ is a smooth submanifold of dimension k,
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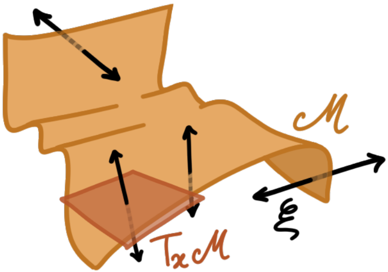
then $\xi \in \mathbf{Tan}^0_\mu$ if and only if (ξ is centred and)

$$v \perp \mathrm{T}_x \mathcal{M} \qquad \xi - \text{almost everywhere.}$$



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Nonsmooth surfaces: Zajíček's theorem

Definition A set $A \subset \mathbb{R}^d$ is DC_k (Difference of Convex of dim k) if up to permuting the axes,

$$A = \left\{ (x_1, \cdots, x_k, \Phi(x_1, \cdots, x_k)) \;\middle|\; \Phi : \mathbb{R}^k \to \mathbb{R}^{d-k}, \text{ with each } \Phi_i = \text{convex} - \text{convex} \right\}.$$

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Theorem 1 of [Zaj79][1] If $\varphi : \mathbb{R}^d \to \mathbb{R}$ is convex, then each $J_k(\varphi)$ is $\sigma-\mathrm{DC_k}$.
Conversely, if $A \subset \mathbb{R}^d$ is $\sigma-\mathrm{DC_k}$, there exists a convex $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that $A \subset J_k(\varphi)$.

[1] L. Zajíček, "On the differentiation of convex functions in finite and infinite dimensional spaces" (1979).
See also G. Alberti, "On the structure of singular sets of convex functions" (1994).

Corollary: sharp answer to Monge's problem

Theorem – Brenier[1]-McCann[2][-Gigli[3]] Let $\mu \in \mathscr{P}_2(\Omega)$.

[1] Y. Brenier, "Polar factorization and monotone rearrangement of vector-valued functions" (1991).
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With the previous notations,

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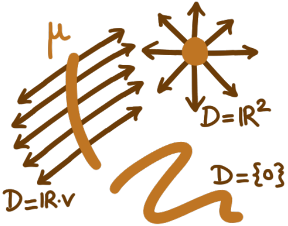
$$\xi \in A \qquad \Longleftrightarrow \qquad [\xi \text{ is centred and } v \in D(x) \text{ for } \xi-\text{almost any } (x,v).]$$



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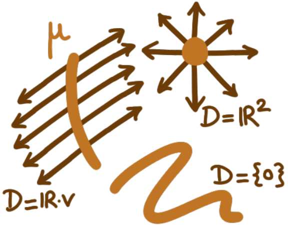

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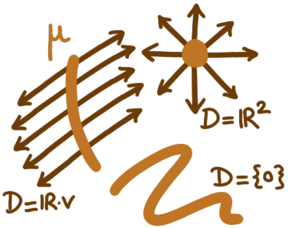

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- By [Gig08], $\mathbf{Tan}^0_\mu$ is a closed convex cone of centred fields.

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- the application D characterizing $\mathbf{Tan}^0_\mu$ is given by $D(x) = (\mathrm{T}_x\, A_k)^\perp$ for μ_k-a.e. $x \in \Omega$.

Decomposition along the direction of splitting

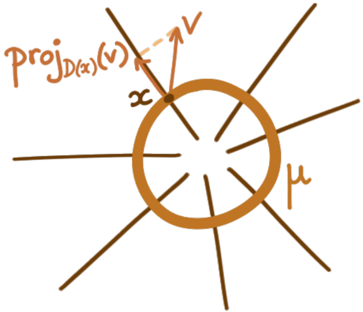
Theorem Let $\mu \in \mathscr{P}_2(\Omega)$. There exists a unique decomposition $\mu = \sum_{k=0}^{d} \mu_k$ in mutually singular measures such that

- μ_k is concentrated on a $\sigma-$DC$_\mathsf{k}$ set A_k, and gives 0 mass to DC$_j$ sets for $j < k$;
- the application D characterizing $\mathbf{Tan}^0_\mu$ is given by $D(x) = (\mathrm{T}_x\, A_k)^\perp$ for μ_k-a.e. $x \in \Omega$.

Explicitly, $\xi \in \mathbf{Tan}^0_\mu$ if and only if ξ is (centred and) concentrated on the normal spaces to each A_k.

Small application: projection on $\mathbf{Tan}_\mu^0$

For each x, denote $\mathrm{proj}_{D(x)} : \mathbb{R}^d \to \mathbb{R}^d$ the projection over $D(x)$.



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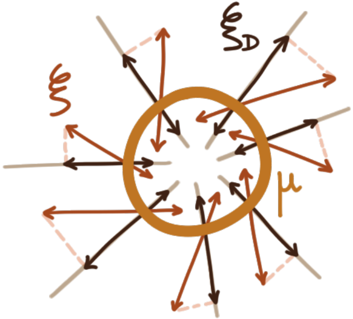
Corollary For any $\xi \in \mathscr{P}_2(\mathrm{T}\,\Omega)_\mu^0$, the measure field

$$\xi_D \coloneqq (\pi_x, \mathrm{proj}_{D(x)}(\pi_v))_\# \xi$$

is the unique minimizer of $W_\mu(\zeta, \xi)$ over $\zeta \in \mathbf{Tan}_\mu^0$.



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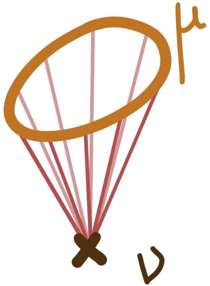

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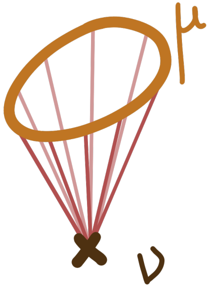

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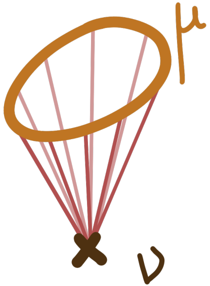

Thank you for your attention!