Befriending $\mathscr{P}_2(\mathbb{R}^d)$

Viscosity solutions of centralized control problems in measure spaces

Averil Prost (LMI, INSA Rouen Normandie)

Joint work with
Hasnaa Zidani (LMI, INSA Rouen Normandie)
Othmane Jerhaoui (IRMAR, INSA Rennes)

December 5, 2023
Workshop Optimal Control, Valparaíso

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Motivation

- Hamilton-Jacobi-Bellman allow to solve *control problems*.

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Motivation

- Hamilton-Jacobi-Bellman allow to solve *control problems*.
- Measure spaces naturally model uncertainty.

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Motivation

- Hamilton-Jacobi-Bellman allow to solve *control problems*.
- Measure spaces naturally model uncertainty.
- Very adapted to concentrations phenomena.

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Motivation

- Hamilton-Jacobi-Bellman allow to solve *control problems*.
- Measure spaces naturally model uncertainty.
- Very adapted to concentrations phenomena.
- Applications in Mean Field Control theory, control of crowd motion, resource allocation, image processing...

Setting

We are interested into Hamilton-Jacobi-Bellman equations of the form

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \qquad \text{(HJB)}$$

in measure spaces, using appropriate *viscosity solutions* [CIL92, BC97].

Motivation

- Hamilton-Jacobi-Bellman allow to solve *control problems*.
- Measure spaces naturally model uncertainty.
- Very adapted to concentrations phenomena.
- Applications in Mean Field Control theory, control of crowd motion, resource allocation, image processing...

Let us explain each term.

The Wasserstein space

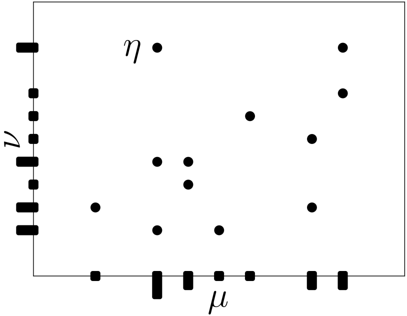
Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures.

The Wasserstein space

Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures. We denote

$\Gamma(\mu, \nu)$ the set of transport plans

$$\left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \begin{matrix} \pi_x \# \eta = \mu, \\ \pi_y \# \eta = \nu \end{matrix} \right\},$$



The Wasserstein space

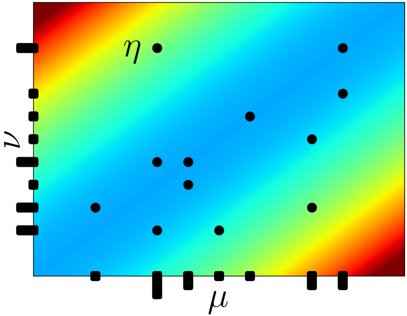
Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures. We denote

$\Gamma(\mu,\nu)$ the set of transport plans

$$\left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \begin{array}{c} \pi_x \# \eta = \mu, \\ \pi_y \# \eta = \nu \end{array} \right\},$$

$d_{\mathcal{W}}^2(\mu,\nu)$ the (squared) distance

$$\inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^2 \, d\eta(x,y).$$



The Wasserstein space

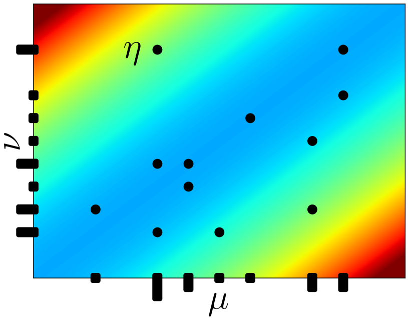
Let $\mu, \nu \in \mathscr{P}(\mathbb{R}^d)$ be two probability measures. We denote

$\Gamma(\mu,\nu)$ the set of transport plans

$$\left\{ \eta \in \mathscr{P}((\mathbb{R}^d)^2) \;\middle|\; \begin{array}{c} \pi_x \# \eta = \mu, \\ \pi_y \# \eta = \nu \end{array} \right\},$$

$d^2_{\mathcal{W}}(\mu,\nu)$ the (squared) distance

$$\inf_{\eta \in \Gamma(\mu,\nu)} \int_{(x,y)} |x-y|^2 \, d\eta(x,y).$$



Def 1 We call **Wasserstein space** the set $\mathscr{P}_2(\mathbb{R}^d)$ given by $\{\mu \in \mathscr{P}(\mathbb{R}^d) \mid d_{\mathcal{W}}(\mu,\delta_0) < \infty\}$, endowed with the distance $d_{\mathcal{W}}$.

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F\left(x, \nabla V(x)\right) = 0$ for all $x \in \mathbb{R}^d$.

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F(x, \nabla V(x)) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F\left(x, \nabla V(x)\right) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.
- If $\nabla V(x)$ does not exist, weaken by using sub/superdifferentials:

$$F(x, p_+) \leqslant 0 \quad \forall p_+ \in \partial_x^+ V, \qquad F(x, p_-) \geqslant 0 \quad \forall p_- \in \partial_x^- V.$$

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F\left(x, \nabla V(x)\right) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.
- If $\nabla V(x)$ does not exist, weaken by using sub/superdifferentials:

$$F(x, p_+) \leqslant 0 \quad \forall p_+ \in \partial_x^+ V, \qquad F(x, p_-) \geqslant 0 \quad \forall p_- \in \partial_x^- V.$$

- These semidifferentials may be represented as gradients of smooth functions that "touch" V from above or below.

Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F(x, \nabla V(x)) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.
- If $\nabla V(x)$ does not exist, weaken by using sub/superdifferentials:

$$F(x, p_+) \leqslant 0 \quad \forall p_+ \in \partial_x^+ V, \qquad F(x, p_-) \geqslant 0 \quad \forall p_- \in \partial_x^- V.$$

- These semidifferentials may be represented as gradients of smooth functions that "touch" V from above or below.

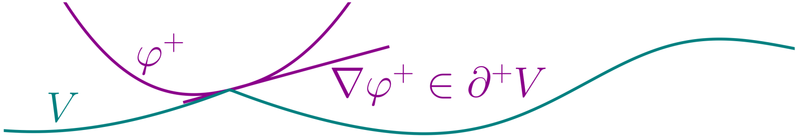
Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F(x, \nabla V(x)) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.
- If $\nabla V(x)$ does not exist, weaken by using sub/superdifferentials:

$$F(x, p_+) \leqslant 0 \quad \forall p_+ \in \partial_x^+ V, \qquad F(x, p_-) \geqslant 0 \quad \forall p_- \in \partial_x^- V.$$

- These semidifferentials may be represented as gradients of smooth functions that "touch" V from above or below.



Viscosity solutions in less than one minute

Consider $\mathbb{R}^d$ as a state space, and an equation $F(x, \nabla V(x)) = 0$.

- Classical $\mathcal{C}^1$ solution: $F(x, \nabla V(x)) = 0$ for all $x \in \mathbb{R}^d$.
- Equivalent to imposing $F(x, \nabla V(x)) \leqslant 0$ and $F(x, \nabla V(x)) \geqslant 0$.
- If $\nabla V(x)$ does not exist, weaken by using sub/superdifferentials:

$$F(x, p_+) \leqslant 0 \quad \forall p_+ \in \partial_x^+ V, \qquad F(x, p_-) \geqslant 0 \quad \forall p_- \in \partial_x^- V.$$

- These semidifferentials may be represented as gradients of smooth functions that "touch" V from above or below.

Trends in the literature

Objective Introduce a simple formalism for viscosity solutions of HJB equations in the Wasserstein space.

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \quad \text{(HJB)}$$

Trends in the literature

Objective Introduce a simple formalism for viscosity solutions of HJB equations in the Wasserstein space.

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \quad \text{(HJB)}$$

Wide literature on this topic and related problems.

- Most often *semidifferentials* ([CQ08, GNT08, MQ18, GT19, JMQ20]),

Trends in the literature

Objective Introduce a simple formalism for viscosity solutions of HJB equations in the Wasserstein space.

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \quad \text{(HJB)}$$

Wide literature on this topic and related problems.

- Most often *semidifferentials* ([CQ08, GNT08, MQ18, GT19, JMQ20]),
- *Test functions* evoqued in [Jim21], promising approach owing to mollification procedures [CGK$^+$22, CM23],

Trends in the literature

Objective Introduce a simple formalism for viscosity solutions of HJB equations in the Wasserstein space.

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \quad \text{(HJB)}$$

Wide literature on this topic and related problems.

- Most often *semidifferentials* ([CQ08, GNT08, MQ18, GT19, JMQ20]),
- *Test functions* evoqued in [Jim21], promising approach owing to mollification procedures [CGK$^+$22, CM23],
- For Eikonal-type equations, *metric viscosity* techniques using metric slopes [AF14, HK15, GŚ15].

Trends in the literature

Objective Introduce a simple formalism for viscosity solutions of HJB equations in the Wasserstein space.

$$-\partial_t V(t,\mu) + H\left(\mu, D_\mu V(t,\mu)\right) = 0, \qquad V(T,\mu) = \mathfrak{J}(\mu) \quad \text{(HJB)}$$

Wide literature on this topic and related problems.

- Most often *semidifferentials* ([CQ08, GNT08, MQ18, GT19, JMQ20]),
- *Test functions* evoqued in [Jim21], promising approach owing to mollification procedures [CGK$^+$22, CM23],
- For Eikonal-type equations, *metric viscosity* techniques using metric slopes [AF14, HK15, GŚ15].

In this talk, we follow a line opened in [JJZ, Jer22, JPZ23].

Table of Contents

Map of directional derivatives

Problem How to define $D_\mu V(t,\mu)$ when μ is a measure?

Map of directional derivatives

Problem How to define $D_\mu V(t,\mu)$ when μ is a measure?

- Huge literature on the topic (Lions lift, Wasserstein gradient...)

Map of directional derivatives

Problem How to define $D_\mu V(t, \mu)$ when μ is a measure?

- Huge literature on the topic (Lions lift, Wasserstein gradient...)
- Functions admitting such gradients are **quite rare** in $\mathscr{P}_2(\mathbb{R}^d)$.

Map of directional derivatives

Problem How to define $D_\mu V(t,\mu)$ when μ is a measure?

- Huge literature on the topic (Lions lift, Wasserstein gradient...)
- Functions admitting such gradients are **quite rare** in $\mathscr{P}_2(\mathbb{R}^d)$.
- In the derivation of HJB equations, enough to consider the variation of the value function along some *directions.*

Map of directional derivatives

Problem How to define $D_\mu V(t,\mu)$ when μ is a measure?

- Huge literature on the topic (Lions lift, Wasserstein gradient...)
- Functions admitting such gradients are **quite rare** in $\mathscr{P}_2(\mathbb{R}^d)$.
- In the derivation of HJB equations, enough to consider the variation of the value function along some *directions.*

Idea: define $D_\mu V(t,\mu)$ as a **map on a set of directions**. Generalizes

$$q \mapsto \langle \nabla_x V(t,x), q \rangle ,$$

but may be nonlinear.

Map of directional derivatives

Problem How to define $D_\mu V(t,\mu)$ when μ is a measure?

- Huge literature on the topic (Lions lift, Wasserstein gradient...)
- Functions admitting such gradients are **quite rare** in $\mathscr{P}_2(\mathbb{R}^d)$.
- In the derivation of HJB equations, enough to consider the variation of the value function along some *directions*.

Idea: define $D_\mu V(t,\mu)$ as a **map on a set of directions**. Generalizes

$$q \mapsto \langle \nabla_x V(t,x), q \rangle ,$$

but may be nonlinear. Then what are admissible "tangent" directions?

Naive definition

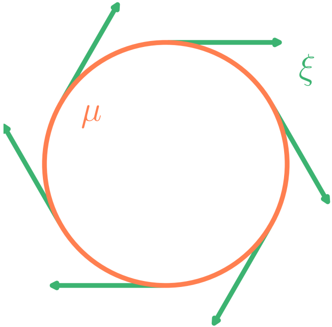
Broad idea Assign to every point x a rule to distribute its mass.

Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$

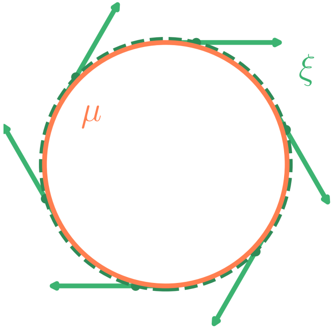


Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$

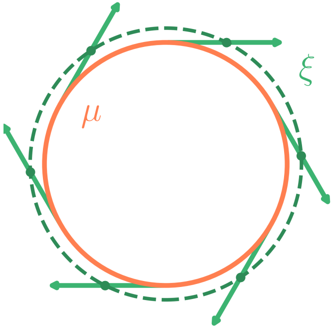


Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu (h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$

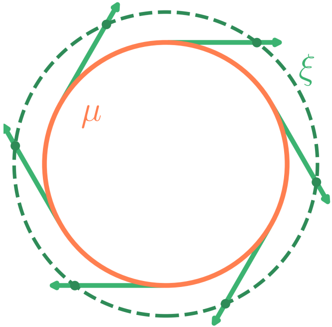


Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu (h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$

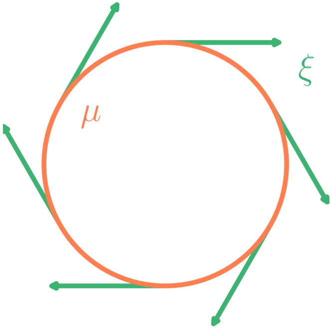


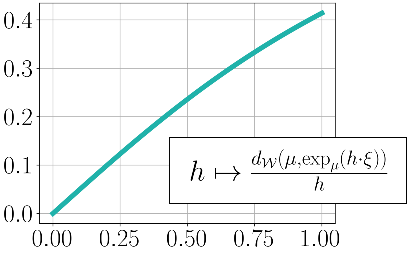
Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$



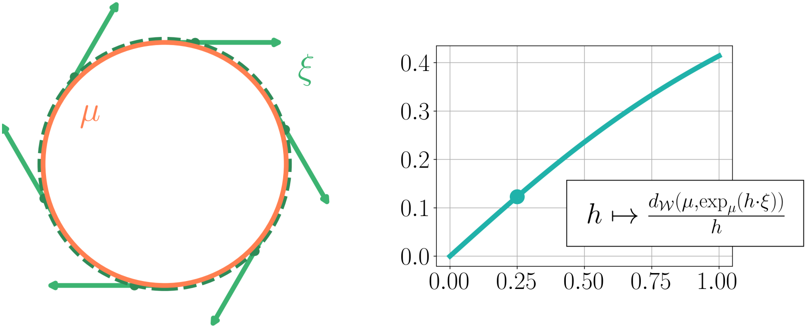


Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu (h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$

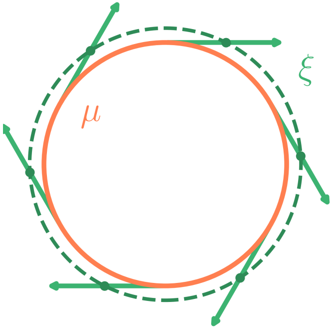


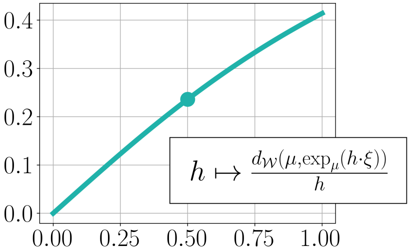
Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$



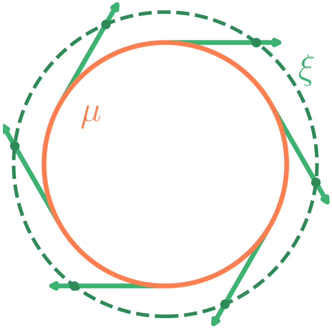


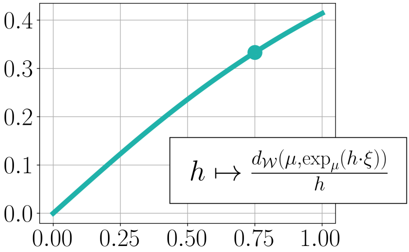
Naive definition

Broad idea Assign to every point x a rule to distribute its mass.

Let $\xi \in \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ a probability on the pairs (x, v) such that v is a tangent vector in $\mathsf{T}_x\mathbb{R}^d$. We denote

$$\exp_\mu(h \cdot \xi) := (\pi_x + h\pi_v)\#\xi \in \mathscr{P}_2(\mathbb{R}^d).$$





Tangent cones [Gig08]

Aim retain only meaningful directions.

Tangent cones [Gig08]

Aim retain only meaningful directions.

Two propositions in the literature: the *regular tangent cone*,

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^1(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d;\mathsf{T}\mathbb{R}^d)},$$

Tangent cones [Gig08]

Aim retain only meaningful directions.

Two propositions in the literature: the *regular tangent cone*,

$$\mathrm{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^1(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d;\mathsf{T}\mathbb{R}^d)},$$

and the *general tangent cone*,

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\left\{\alpha\cdot\xi \;\middle|\; \alpha \in \mathbb{R}^+,\ h \mapsto \exp_\mu(h\cdot\xi) \text{ is a geodesic}\right\}}^{W_\mu}.$$

Here W_μ is a generalization of the L^2_μ−distance to $\mathscr{P}_2(\mathsf{T}\mathbb{R}^d)_\mu$.

Tangent cones [Gig08]

Aim retain only meaningful directions.

Two propositions in the literature: the *regular tangent cone*,

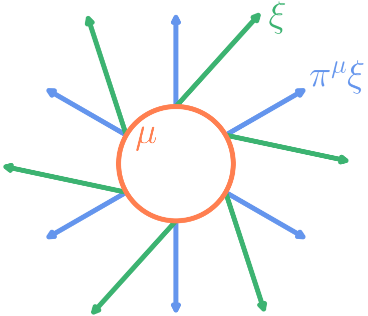
$$\mathsf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\{\nabla\varphi \mid \varphi \in \mathcal{C}_c^1(\mathbb{R}^d)\}}^{L^2_\mu(\mathbb{R}^d;\mathsf{T}\mathbb{R}^d)},$$

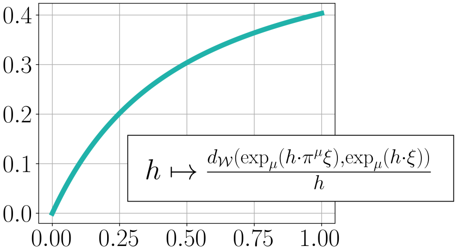
and the *general tangent cone*,

$$\mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) := \overline{\left\{\alpha \cdot \xi \;\middle|\; \alpha \in \mathbb{R}^+,\ h \mapsto \exp_\mu(h \cdot \xi) \text{ is a geodesic}\right\}}^{W_\mu}.$$

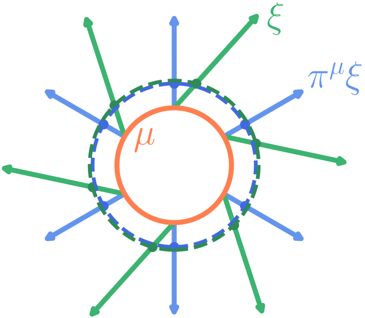
Here W_μ is a generalization of the $L^2_\mu-$distance to $\mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$. Both tangent cones enjoy a well-defined *projection mapping*, and we denote $\pi^\mu : \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu \to \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d)$ the projection on $\mathbf{Tan}_\mu$.

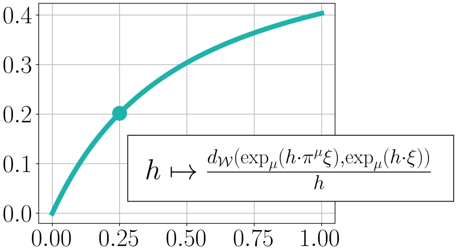
Example


ξ
$\pi^\mu \xi$
μ

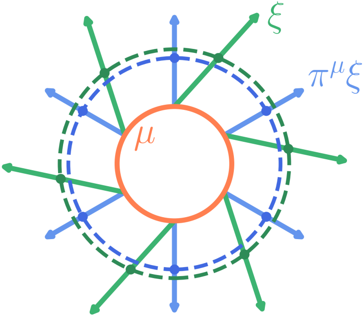

0.4
0.3
0.2
0.1
0.0
0.00
0.25
0.50
0.75
1.00
$h \mapsto \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \pi^\mu \xi), \exp_\mu(h \cdot \xi))}{h}$

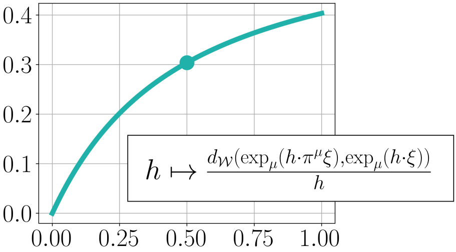
Example


ξ
$\pi^\mu \xi$
μ

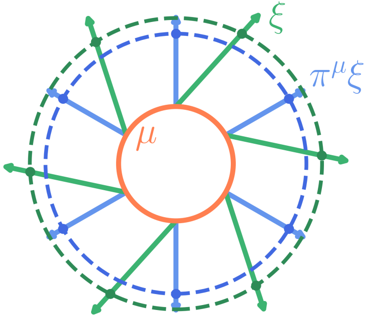

0.4
0.3
0.2
0.1
0.0
0.00
0.25
0.50
0.75
1.00
$h \mapsto \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \pi^\mu \xi), \exp_\mu(h \cdot \xi))}{h}$

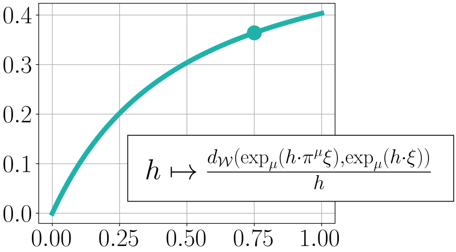
Example


ξ
$\pi^\mu \xi$
μ

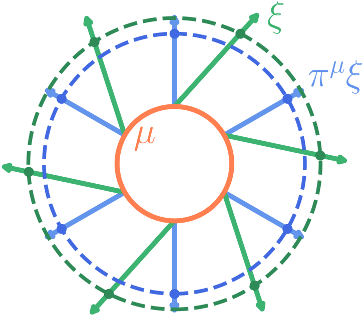

0.4
0.3
0.2
0.1
0.0
0.00
0.25
0.50
0.75
1.00
$h \mapsto \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \pi^\mu \xi), \exp_\mu(h \cdot \xi))}{h}$

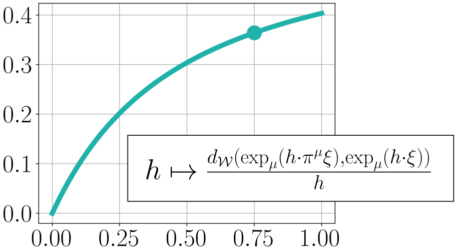
Example


ξ
$\pi^\mu \xi$
μ


0.4
0.3
0.2
0.1
0.0
0.00
0.25
0.50
0.75
1.00
$h \mapsto \frac{d_{\mathcal{W}}(\exp_\mu(h \cdot \pi^\mu \xi), \exp_\mu(h \cdot \xi))}{h}$

Example





Theorem In general, there holds

$$d_{\mathcal{W}}\left(\exp_\mu(h\cdot\pi^\mu\xi),\exp_\mu(h\cdot\xi)\right) = o(h).$$

Consequently, $D_\mu\varphi(\mu)(\xi) = D_\mu\varphi(\mu)(\pi^\mu\xi)$ whenever φ is Lipschitz.

The metric cotangent bundle

We now wish to measure variations of maps along directions in $\mathbf{Tan}_\mu$.

The metric cotangent bundle

We now wish to measure variations of maps along directions in $\mathbf{Tan}_\mu$.

Def 2 – Metric cotangent bundle Let $\mathbb{T} = \bigcup_\mu \{\mu\} \times \mathbb{T}_\mu$, where

$$\mathbb{T}_\mu := \left\{ p : \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \;\middle|\; \begin{array}{c} p \text{ is Lipschitz in } W_\mu \text{ and} \\ p(\lambda\xi) = \lambda p(\xi) \quad \forall \lambda \geqslant 0 \end{array} \right\}.$$

The metric cotangent bundle

We now wish to measure variations of maps along directions in $\mathbf{Tan}_\mu$.

Def 2 – Metric cotangent bundle Let $\mathbb{T} = \bigcup_\mu \{\mu\} \times \mathbb{T}_\mu$, where

$$\mathbb{T}_\mu := \left\{ p : \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R} \;\middle|\; \begin{array}{c} p \text{ is Lipschitz in } W_\mu \text{ and} \\ p(\lambda\xi) = \lambda p(\xi) \quad \forall \lambda \geqslant 0 \end{array} \right\}.$$

Elements of $\mathbb{T}$ replace linear mappings as the elementary model for infinitesimal approximation of sufficiently smooth functions. Indeed,

$$\xi \in \mathbf{Tan}_\mu \mathscr{P}_2(\mathbb{R}^d) \mapsto D_\mu \varphi(\xi) := \lim_{h \searrow 0} \frac{\varphi((\pi_x + h\pi_v)\#\xi) - \varphi(\mu)}{h}$$

belongs to $\mathbb{T}$ whenever φ is locally Lipschitz and the limits exist.

Table of Contents

Controlled continuity equations

Consider

- $U \subset \mathbb{R}^\kappa$ a compact set of controls, and $u \in L^0([0,T];U)$;

Controlled continuity equations

Consider

- $U \subset \mathbb{R}^\kappa$ a compact set of controls, and $u \in L^0([0,T];U)$;
- a dynamic $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to \mathcal{C}(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$ uniformly Lipschitz,

Controlled continuity equations

Consider

- $U \subset \mathbb{R}^\kappa$ a compact set of controls, and $u \in L^0([0,T];U)$;
- a dynamic $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to \mathcal{C}(\mathbb{R}^d; \mathsf{T}\mathbb{R}^d)$ uniformly Lipschitz,
- a final time $T > 0$, and an initial point $\nu \in \mathscr{P}_2(\mathbb{R}^d)$.

Controlled continuity equations

Consider

- $U \subset \mathbb{R}^\kappa$ a compact set of controls, and $u \in L^0([0,T];U)$;
- a dynamic $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to \mathcal{C}(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$ uniformly Lipschitz,
- a final time $T > 0$, and an initial point $\nu \in \mathscr{P}_2(\mathbb{R}^d)$.

The *controlled continuity equation* writes

$$\partial_t \mu_t + \mathrm{div}\,(f[\mu_t, u_t]\#\mu_t) = 0, \qquad \mu_0 = \nu. \tag{CE}$$

Controlled continuity equations

Consider

- $U \subset \mathbb{R}^\kappa$ a compact set of controls, and $u \in L^0([0,T];U)$;
- a dynamic $f : \mathscr{P}_2(\mathbb{R}^d) \times U \to \mathcal{C}(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$ uniformly Lipschitz,
- a final time $T > 0$, and an initial point $\nu \in \mathscr{P}_2(\mathbb{R}^d)$.

The *controlled continuity equation* writes

$$\partial_t \mu_t + \mathrm{div}\,(f[\mu_t, u_t]\#\mu_t) = 0, \qquad \mu_0 = \nu. \tag{CE}$$

Proposition – Well-posedness and Filippov Theorem ([BF23])
Each $u \in L^0([0,T];U)$ generates an unique solution of (CE). If $\{f[\mu,u] \mid u \in U\}$ is convex in $\mathcal{C}(\mathbb{R}^d; \mathrm{T}\mathbb{R}^d)$, the set of solutions associated to controls in $L^0([0,T];U)$ is compact in $\mathcal{C}([0,T]; \mathscr{P}_2(\mathbb{R}^d))$.

The control problem

Given $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ an initial measure, we consider the problem

$$\text{Minimize} \quad \mathfrak{J}\left(\mu_T\right) \text{ such that } (\mu_s)_{s\in[0,T]} \text{ solves (CE) with } \mu_0 = \nu.$$

The control problem

Given $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ an initial measure, we consider the problem

$$\text{Minimize} \quad \mathfrak{J}\left(\mu_T\right) \text{ such that } (\mu_s)_{s\in[0,T]} \text{ solves (CE) with } \mu_0 = \nu.$$

Def 3 – Value function Denote $V : [0,T]\times\mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ the map

$$V(t,\nu) := \inf\left\{\mathfrak{J}(\mu_T) \;\middle|\; (\mu_s)_{s\in[t,T]} \text{ solves (CE) with } \mu_t = \nu\right\}$$

The control problem

Given $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ an initial measure, we consider the problem

$$\text{Minimize} \quad \mathfrak{J}(\mu_T) \text{ such that } (\mu_s)_{s\in[0,T]} \text{ solves (CE) with } \mu_0 = \nu.$$

Def 3 – Value function Denote $V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ the map

$$V(t,\nu) := \inf\left\{\mathfrak{J}(\mu_T) \;\middle|\; (\mu_s)_{s\in[t,T]} \text{ solves (CE) with } \mu_t = \nu\right\}$$

Under our assumptions, V is Lipschitz-continuous in time and measure.

The control problem

Given $\nu \in \mathscr{P}_2(\mathbb{R}^d)$ an initial measure, we consider the problem

$$\text{Minimize} \quad \mathfrak{J}\left(\mu_T\right) \text{ such that } (\mu_s)_{s\in[0,T]} \text{ solves (CE) with } \mu_0 = \nu.$$

Def 3 – Value function Denote $V : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ the map

$$V(t,\nu) := \inf\left\{\mathfrak{J}(\mu_T) \;\middle|\; (\mu_s)_{s\in[t,T]} \text{ solves (CE) with } \mu_t = \nu\right\}$$

Under our assumptions, V is Lipschitz-continuous in time and measure.

Proposition V satisfies the dynamical programming principle

$$V(t,\nu) = \inf\left\{V(t+h, \mu_{t+h}) \;\middle|\; (\mu_s)_{s\in[t,t+h]} \text{ solves (CE) with } \mu_t = \nu\right\}.$$

The control Hamiltonian

Let now $F : \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ be a dynamic equal to

$$F[\mu] := \{f[\mu, u]\#\mu \mid u \in U\}.$$

The control Hamiltonian

Let now $F : \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ be a dynamic equal to

$$F[\mu] \coloneqq \{f[\mu, u]\#\mu \mid u \in U\}.$$

In the classical case,

$$H_{\mathbb{R}^d}(x, p) \coloneqq \sup_{u \in U} -\langle p, f(x, u)\rangle$$

The control Hamiltonian

Let now $F : \mathscr{P}_2(\mathbb{R}^d) \rightrightarrows \mathscr{P}_2(\mathsf{T}\mathbb{R}^d)$ be a dynamic equal to

$$F[\mu] \coloneqq \{f[\mu,u]\#\mu \mid u \in U\}\,.$$

In the classical case,

$$H_{\mathbb{R}^d}(x,p) \coloneqq \sup_{u\in U} - \langle p, f(x,u)\rangle$$

Def 4 – Hamiltonian in (HJB) We consider $H : \mathbb{T} \to \mathbb{R}$ given by

$$H(\mu,p) \coloneqq \sup_{u\in U} -p\left(\pi^{\mu} f[\mu,u]\#\mu\right) = \sup_{\xi\in F[\mu]} -p\left(\pi^{\mu}\xi\right).$$

Test functions

The principle of our definition is to report directional derivatives of viscosity solutions on sufficiently smooth test functions.

Test functions

The principle of our definition is to report directional derivatives of viscosity solutions on sufficiently smooth test functions.

Def 5 – Test functions Define $\mathscr{T}_\pm$ as the set of all applications $\varphi :]0, T[\times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

- φ is locally Lipschitz,

Test functions

The principle of our definition is to report directional derivatives of viscosity solutions on sufficiently smooth test functions.

Def 5 – Test functions Define $\mathscr{T}_{\pm}$ as the set of all applications $\varphi :]0,T[\times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

- φ is locally Lipschitz,
- for all $t \in]0,T[$, $\pm\varphi(t,\cdot)$ is semiconcave,

Test functions

The principle of our definition is to report directional derivatives of viscosity solutions on sufficiently smooth test functions.

Def 5 – Test functions Define $\mathscr{T}_{\pm}$ as the set of all applications $\varphi :]0,T[\times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ such that

- φ is locally Lipschitz,
- for all $t \in]0,T[$, $\pm\varphi(t,\cdot)$ is semiconcave,
- the application $(t,\mu) \mapsto \partial_t \varphi(t,\mu)$ is defined and locally Lipschitz.

Regularity

Let (Y, d) be a complete metric space.

Def 6 – Local uniform upper semicontinuity

Regularity

Let (Y, d) be a complete metric space.

Def 6 – Local uniform upper semicontinuity An application $v : Y \to \mathbb{R}$ is locally uniformly upper semicontinuous (luusc) if the map

$$B \mapsto \sup_{y \in B} v(y)$$

is locally upper semicontinuous in the space of nonempty, bounded and closed sets of Y endowed with the Hausdorff distance.

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous,

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous, satisfies $v(T,\mu) \leqslant \mathfrak{J}(\mu)$,

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous, satisfies $v(T,\mu) \leqslant \mathfrak{J}(\mu)$, and if whenever $u - \varphi$ reaches a maximum in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_+$,

$$-\partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \leqslant 0.$$

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous, satisfies $v(T,\mu) \leqslant \mathfrak{J}(\mu)$, and if whenever $u - \varphi$ reaches a maximum in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_+$,

$$-\partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \leqslant 0.$$

- viscosity supersolution of (HJB) if it is locally uniformly lower semicontinuous,

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous, satisfies $v(T,\mu) \leqslant \mathfrak{J}(\mu)$, and if whenever $u - \varphi$ reaches a maximum in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_+$,

$$-\partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \leqslant 0.$$

- viscosity supersolution of (HJB) if it is locally uniformly **lower** semicontinuous, satisfies $v(T,\mu) \geqslant \mathfrak{J}(\mu)$ and if whenever $u - \varphi$ reaches a **minimum** in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_-$,

$$-\partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \geqslant 0.$$

Definition

Def 7 – Viscosity solutions A map $v : [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is a

- viscosity subsolution of (HJB) if it is locally uniformly upper semicontinuous, satisfies $v(T,\mu) \leqslant \mathfrak{J}(\mu)$, and if whenever $u - \varphi$ reaches a maximum in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_+$,

$$- \partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \leqslant 0.$$

- viscosity supersolution of (HJB) if it is locally uniformly **lower** semicontinuous, satisfies $v(T,\mu) \geqslant \mathfrak{J}(\mu)$ and if whenever $u - \varphi$ reaches a **minimum** in $(t,\mu) \in]0,T[\times \mathscr{P}_2(\mathbb{R}^d)$ for some $\varphi \in \mathscr{T}_-$,

$$- \partial_t \varphi(t,\mu) + H\left(\mu, D_\mu \varphi(t,\mu)\right) \geqslant 0.$$

- viscosity solution is it is both a sub and supersolution of (HJB).

Table of Contents

The value function is a viscosity solution

Recall that the value function of our control problem is defined as

$$V(t,\nu) \coloneqq \inf \left\{ \mathfrak{J}(\mu_T) \;\middle|\; (\mu_s)_{s\in[t,T]} \text{ solves (CE)}, \text{ and } \mu_t = \nu \right\}.$$

The value function is a viscosity solution

Recall that the value function of our control problem is defined as

$$V(t,\nu) \coloneqq \inf \left\{ \mathfrak{J}(\mu_T) \;\middle|\; (\mu_s)_{s\in[t,T]} \text{ solves (CE)}, \text{ and } \mu_t = \nu \right\}.$$

Theorem 1 Assume that the dynamic f is Lipschitz-continuous and that $\mathfrak{J}$ is locally uniformly continuous. Then the value function V is a viscosity solution of the Hamilton-Jacobi-Bellman equation (HJB) in the sense of Def 7.

Uniqueness

Theorem 2 – Comparison principle Let v be a viscosity subsolution and w a viscosity supersolution of (HJB). Then

$$\sup_{(t,\mu)\in]0,T]\times \mathscr{P}_2(\mathbb{R}^d)} v(t,\mu) - w(t,\mu) \leqslant 0.$$

Uniqueness

Theorem 2 – Comparison principle Let v be a viscosity subsolution and w a viscosity supersolution of (HJB). Then

$$\sup_{(t,\mu)\in]0,T]\times\mathscr{P}_2(\mathbb{R}^d)} v(t,\mu) - w(t,\mu) \leqslant 0.$$

The proof is inspired from [FGŚ17] in Hilbert spaces.
Consequently, the value function is the *unique* viscosity solution of (HJB).

Conclusion and perspectives

- Definition of viscosity solutions that uses the ambient geometry.

Conclusion and perspectives

- Definition of viscosity solutions that uses the ambient geometry.
- Retrieves the comparison principle and the link with control problems.

Conclusion and perspectives

- Definition of viscosity solutions that uses the ambient geometry.
- Retrieves the comparison principle and the link with control problems.

Perspective in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$:

- weaken the topology to render $\mathscr{P}_2(\mathbb{R}^d)$ locally compact (ongoing work with Cristopher Hermosilla).

Conclusion and perspectives

- Definition of viscosity solutions that uses the ambient geometry.
- Retrieves the comparison principle and the link with control problems.

Perspective in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$:

- weaken the topology to render $\mathscr{P}_2(\mathbb{R}^d)$ locally compact (ongoing work with Cristopher Hermosilla).
- Clarify the links with "geometrical" notions of viscosity solutions using semidifferentials.

Conclusion and perspectives

- Definition of viscosity solutions that uses the ambient geometry.
- Retrieves the comparison principle and the link with control problems.

Perspective in the Wasserstein space $\mathscr{P}_2(\mathbb{R}^d)$:

- weaken the topology to render $\mathscr{P}_2(\mathbb{R}^d)$ locally compact (ongoing work with Cristopher Hermosilla).
- Clarify the links with "geometrical" notions of viscosity solutions using semidifferentials.

Out of $\mathbb{R}^d$:

- Determine in which conditions the squared Wasserstein distance is directionally differentiable if the underlying space is a 1-dimensional network.

Thank you!

[AF14] Luigi Ambrosio and Jin Feng.
On a class of first order Hamilton–Jacobi equations in metric spaces.
Journal of Differential Equations, 256(7):2194–2245, April 2014.

[BC97] Martino Bardi and Italo Capuzzo-Dolcetta.
Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations.
Birkhäuser Boston, Boston, MA, 1997.

[BF23] Benoît Bonnet and Hélène Frankowska.
Caratheodory Theory and A Priori Estimates for Continuity Inclusions in the Space of Probability Measures, May 2023.

[CGK+22] Andrea Cosso, Fausto Gozzi, Idris Kharroubi, Huyên Pham, and Mauro Rosestolato.
Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions, February 2022.

[CIL92] Michael G. Crandall, Hitoshi Ishii, and Pierre-Louis Lions.
User's guide to viscosity solutions of second order partial differential equations, June 1992.

[CM23] Andrea Cosso and Mattia Martini.
On smooth approximations in the Wasserstein space, March 2023.

[CQ08] P. Cardaliaguet and M. Quincampoix.
Deterministic differential games under probability knowledge of initial condition.
International Game Theory Review, 10(01):1–16, March 2008.

[FGŚ17] Giorgio Fabbri, Fausto Gozzi, and Andrzej Święch.
Stochastic Optimal Control in Infinite Dimension, volume 82 of *Probability Theory and Stochastic Modelling*.
Springer International Publishing, Cham, 2017.

[Gig08] Nicola Gigli.
On the Geometry of the Space of Probability Measures Endowed with the Quadratic Optimal Transport Distance.
PhD thesis, Scuola Normale Superiore di Pisa, Pisa, 2008.

[GNT08] Wilfrid Gangbo, Truyen Nguyen, and Adrian Tudorascu.
Hamilton-Jacobi Equations in the Wasserstein Space.
Methods and Applications of Analysis, 15(2):155–184, 2008.

[GŚ15] Wilfrid Gangbo and Andrzej Święch.
Metric viscosity solutions of Hamilton–Jacobi equations depending on local slopes.
Calculus of Variations and Partial Differential Equations, 54(1):1183–1218, September 2015.

[GT19] Wilfrid Gangbo and Adrian Tudorascu.
On differentiability in the Wasserstein space and well-posedness for Hamilton–Jacobi equations.
Journal de Mathématiques Pures et Appliquées, 125:119–174, May 2019.

[HK15] Ryan Hynd and Hwa Kil Kim.
Value functions in the Wasserstein spaces: Finite time horizons.
Journal of Functional Analysis, 269(4):968–997, August 2015.

[Jer22] Othmane Jerhaoui.
Viscosity Theory of First Order Hamilton Jacobi Equations in Some Metric Spaces.
PhD thesis, Institut Polytechnique de Paris, Paris, 2022.

[Jim21] Chloé Jimenez.
A new proof concerning the Derivative in P2(Rd) & a definition of Viscosity Solutions. (From a preprint in collaboration with A.Marigonda and M. Quincampoix.), December 2021.

[JJZ] Frédéric Jean, Othmane Jerhaoui, and Hasnaa Zidani.
Deterministic optimal control on Riemannian manifolds under probability knowledge of the initial condition.
page 30.

[JMQ20] Chloé Jimenez, Antonio Marigonda, and Marc Quincampoix.
Optimal control of multiagent systems in the Wasserstein space.
Calculus of Variations and Partial Differential Equations, 59, March 2020.

[JPZ23] Othmane Jerhaoui, Averil Prost, and Hasnaa Zidani.
Viscosity solutions of centralized control problems in measure spaces, 2023.

[MQ18] Antonio Marigonda and Marc Quincampoix.
Mayer control problem with probabilistic uncertainty on initial positions.
Journal of Differential Equations, 264(5):3212–3252, March 2018.