Minimalist analysis

A Lagrangian scheme for first-order HJB equations using neural networks

Averil Prost
Joint work with Olivier Bokanowski & Xavier Warin

May 22, 2023
SMAI 2023

Table of Contents

Setting of the problem

Let $T > 0$. We consider the solution $V = V(t,x)$ of

$$\min\Big(-\partial_t V + \max_{a\in A} \langle \nabla V, f(x,a)\rangle, V - g(x)\Big) = 0, \qquad V(T,x) = \max\left(\mathfrak{J}(x), g(x)\right). \quad \text{(HJ)}$$

Setting of the problem

Let $T > 0$. We consider the solution $V = V(t,x)$ of

$$\min\left(-\partial_t V + \max_{a\in A}\langle \nabla V, f(x,a)\rangle, V - g(x)\right) = 0, \qquad V(T,x) = \max\left(\mathfrak{J}(x), g(x)\right). \quad \text{(HJ)}$$

Notations and running assumptions Here

(A1)

- $A \subset \mathbb{R}^\kappa$ is a compact set, and $\mathbb{A}_{[t,T]}$ the set of measurable $a(\cdot) : [t,T] \to A$,
- $f : \mathbb{R}^d \times A \to T\mathbb{R}^d$ is a Lipschitz dynamic such that $f(x,A)$ is convex $\forall x \in \mathbb{R}^d$,
- The obstacle function $g : \mathbb{R}^d \to \mathbb{R}$ and terminal cost $\mathfrak{J} : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz.

Setting of the problem

Let $T > 0$. We consider the solution $V = V(t, x)$ of

$$\min\left(-\partial_t V + \max_{a\in A}\langle \nabla V, f(x,a)\rangle, V - g(x)\right) = 0, \qquad V(T,x) = \max\left(\mathfrak{J}(x), g(x)\right). \quad \text{(HJ)}$$

Notations and running assumptions Here

(A1)

- $A \subset \mathbb{R}^\kappa$ is a compact set, and $\mathbb{A}_{[t,T]}$ the set of measurable $a(\cdot) : [t,T] \to A$,
- $f : \mathbb{R}^d \times A \to T\mathbb{R}^d$ is a Lipschitz dynamic such that $f(x, A)$ is convex $\forall x \in \mathbb{R}^d$,
- The obstacle function $g : \mathbb{R}^d \to \mathbb{R}$ and terminal cost $\mathfrak{J} : \mathbb{R}^d \to \mathbb{R}$ are Lipschitz.

Well-posedness ([ABZ13]) There exists an unique continuous viscosity solution of (HJ).

Origin of the problem (1/3)

Let $f_0 : \mathbb{R}^{d-1} \times A \to T\mathbb{R}^{d-1}$, choose an "admissible" closed set $K \subset \mathbb{R}^{d-1}$ and denote

$$\mathbb{B}_{\xi,[t,T]} := \left\{ a(\cdot) \in \mathbb{A}_{[t,T]} \;\middle|\; \gamma_s^{t,\xi,a} \in K \quad \forall s \in [t,T],\ \dot{\gamma}_s^{t,\xi,a} = f_0(\gamma_s^{t,\xi,a}, a(s)),\ \gamma_t^{t,\xi,a} = \xi \right\}.$$

Origin of the problem (1/3)

Let $f_0 : \mathbb{R}^{d-1} \times A \to T\mathbb{R}^{d-1}$, choose an "admissible" closed set $K \subset \mathbb{R}^{d-1}$ and denote

$$\mathbb{B}_{\xi,[t,T]} := \left\{ a(\cdot) \in \mathbb{A}_{[t,T]} \;\middle|\; \gamma_s^{t,\xi,a} \in K \quad \forall s \in [t,T],\ \dot{\gamma}_s^{t,\xi,a} = f_0(\gamma_s^{t,\xi,a}, a(s)),\ \gamma_t^{t,\xi,a} = \xi \right\}.$$

State-constrained control problem Let $L, J : \mathbb{R}^{d-1} \to \mathbb{R}$ be Lipschitz, and

Find $a^* \in \mathbb{B}_{[t,T]}$ that minimizes $a \mapsto \int_{s=t}^{T} L\left(\gamma_s^{t,\xi,a}\right) ds + J\left(\gamma_T^{t,\xi,a}\right)$ over all $a(\cdot) \in \mathbb{B}_{[t,T]}$.

Origin of the problem (1/3)

Let $f_0 : \mathbb{R}^{d-1} \times A \to T\mathbb{R}^{d-1}$, choose an "admissible" closed set $K \subset \mathbb{R}^{d-1}$ and denote

$$\mathbb{B}_{\xi,[t,T]} := \left\{ a(\cdot) \in \mathbb{A}_{[t,T]} \;\middle|\; \gamma_s^{t,\xi,a} \in K \quad \forall s \in [t,T],\ \dot{\gamma}_s^{t,\xi,a} = f_0(\gamma_s^{t,\xi,a}, a(s)),\ \gamma_t^{t,\xi,a} = \xi \right\}.$$

State-constrained control problem Let $L, J : \mathbb{R}^{d-1} \to \mathbb{R}$ be Lipschitz, and

Find $a^* \in \mathbb{B}_{[t,T]}$ that minimizes $a \mapsto \int_{s=t}^{T} L\left(\gamma_s^{t,\xi,a}\right) ds + J\left(\gamma_T^{t,\xi,a}\right)$ over all $a(\cdot) \in \mathbb{B}_{[t,T]}$.

Introduce the corresponding value function

$$u(t,\xi) := \inf \left\{ \int_{s=t}^{T} L\left(\gamma_s^{t,\xi,a}\right) ds + J\left(\gamma_T^{t,\xi,a}\right) \;\middle|\; a(\cdot) \in \mathbb{B}_{[t,T]} \right\}.$$

Origin of the problem (2/3)

(A2) Assume $K = \{g_0 \leqslant 0\}$, where $g_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz.

Origin of the problem (2/3)

(A2) Assume $K = \{g_0 \leqslant 0\}$, where $g_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz.

Let then

$$x = (\xi, z)\,, \quad f(x,a) := (f_0(\xi,a), L(\xi))\,, \quad g(x) = g_0(\xi), \quad \mathfrak{J}(x) = J(\xi) - z.$$

Origin of the problem (2/3)

(A2) Assume $K = \{g_0 \leqslant 0\}$, where $g_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz.

Let then

$$x = (\xi, z)\,, \quad f(x,a) := (f_0(\xi,a), L(\xi))\,, \quad g(x) = g_0(\xi), \quad \mathfrak{J}(x) = J(\xi) - z.$$

Let again $y_{\cdot}^{t,x,a}$ solve $\dot{y}_s = f(y_x, a(s))$. Introduce the auxilliary map

$$V(t,x) := \inf\left\{\max\left(\mathfrak{J}\left(y_T^{t,x,a}\right), \max_{s\in[t,T]} g\left(y_s^{t,x,a}\right)\right) \,\middle|\, a(\cdot) \in \mathbb{A}_{[t,T]}\right\}.$$

Origin of the problem (2/3)

(A2) Assume $K = \{g_0 \leqslant 0\}$, where $g_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ is Lipschitz.

Let then

$$x = (\xi, z)\,, \quad f(x,a) := (f_0(\xi,a), L(\xi))\,, \quad g(x) = g_0(\xi), \quad \mathfrak{J}(x) = J(\xi) - z.$$

Let again $y^{t,x,a}_{\cdot}$ solve $\dot{y}_s = f(y_x, a(s))$. Introduce the auxilliary map

$$V(t,x) := \inf\left\{ \max\left(\mathfrak{J}\left(y_T^{t,x,a}\right), \max_{s\in[t,T]} g\left(y_s^{t,x,a}\right) \right) \;\middle|\; a(\cdot) \in \mathbb{A}_{[t,T]} \right\}.$$

Link between both ([ABZ13]) The auxilliary map V solves (HJ), and there holds

$$u(t,\xi) = \inf\left\{ z \in \mathbb{R} \mid V(t,(\xi,z)) \leqslant 0 \right\} \qquad \text{(with the convention } \inf(\emptyset) = +\infty.)$$

Origin of the problem (3/3)

Example We want to minimize $\xi \mapsto |\gamma_T^{t,\xi,a}|$, where $\dot{\gamma}_s^{t,\xi,a} = a(s)$ and $|\gamma| \geqslant 1$. Let $A = [-1,1]$, $f_0(\xi,a) \coloneqq a$, $L = 0$, $J(\xi) = |\xi|$ and $g_0(\xi) = 1 - |\xi|$.

Origin of the problem (3/3)

Example We want to minimize $\xi \mapsto |\gamma_T^{t,\xi,a}|$, where $\dot{\gamma}_s^{t,\xi,a} = a(s)$ and $|\gamma| \geqslant 1$. Let $A = [-1,1]$, $f_0(\xi, a) \coloneqq a$, $L = 0$, $J(\xi) = |\xi|$ and $g_0(\xi) = 1 - |\xi|$. The auxilliary problem reads

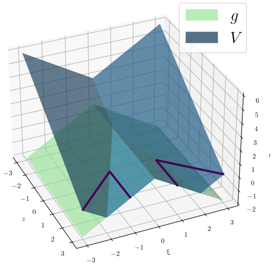
$$V(t,x) = \inf_{a \in \mathbb{A}_{[t,T]}} \left\{ \left| y_T^{t,x,a} \right| \bigvee \max_{s \in [t,T]} \left(1 - \left| y_s^{t,x,a} \right|\right) \right\}, \quad \begin{cases} (-\partial_t V + |\partial_{x_1} V|) \wedge (V - g) = 0, \\ V(T,x) = (|x_1| - x_2) \vee (1 - |x_1|). \end{cases}$$

Origin of the problem (3/3)

Example We want to minimize $\xi \mapsto |\gamma_T^{t,\xi,a}|$, where $\dot{\gamma}_s^{t,\xi,a} = a(s)$ and $|\gamma| \geqslant 1$. Let $A = [-1,1]$, $f_0(\xi,a) := a$, $L = 0$, $J(\xi) = |\xi|$ and $g_0(\xi) = 1 - |\xi|$. The auxilliary problem reads

$$V(t,x) = \inf_{a \in \mathbb{A}_{[t,T]}} \left\{ \left| y_T^{t,x,a} \right| \bigvee \max_{s \in [t,T]} \left(1 - \left| y_s^{t,x,a} \right| \right) \right\}, \quad \begin{cases} (-\partial_t V + |\partial_{x_1} V|) \wedge (V - g) = 0, \\ V(T,x) = (|x_1| - x_2) \vee (1 - |x_1|). \end{cases}$$

Solution V at time $t = T = 1.00$

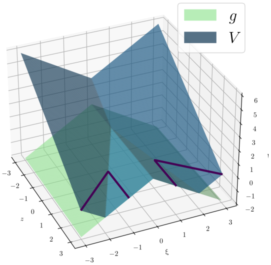


Origin of the problem (3/3)

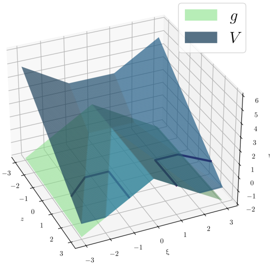
Example We want to minimize $\xi \mapsto |\gamma_T^{t,\xi,a}|$, where $\dot{\gamma}_s^{t,\xi,a} = a(s)$ and $|\gamma| \geqslant 1$. Let $A = [-1,1]$, $f_0(\xi,a) := a$, $L = 0$, $J(\xi) = |\xi|$ and $g_0(\xi) = 1 - |\xi|$. The auxilliary problem reads

$$V(t,x) = \inf_{a \in \mathbb{A}_{[t,T]}} \left\{ \left| y_T^{t,x,a} \right| \bigvee \max_{s \in [t,T]} \left(1 - \left| y_s^{t,x,a} \right| \right) \right\}, \quad \begin{cases} (-\partial_t V + |\partial_{x_1} V|) \wedge (V - g) = 0, \\ V(T,x) = (|x_1| - x_2) \vee (1 - |x_1|). \end{cases}$$

Solution V at time $t = T = 1.00$



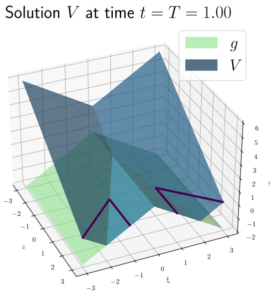
Solution V at time $t = 0.00$

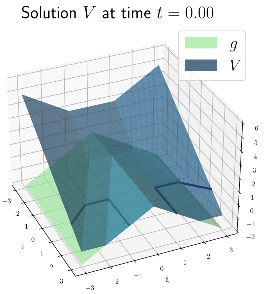


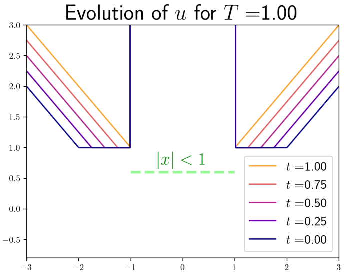
Origin of the problem (3/3)

Example We want to minimize $\xi \mapsto |\gamma_T^{t,\xi,a}|$, where $\dot{\gamma}_s^{t,\xi,a} = a(s)$ and $|\gamma| \geqslant 1$. Let $A = [-1,1]$, $f_0(\xi,a) := a$, $L = 0$, $J(\xi) = |\xi|$ and $g_0(\xi) = 1 - |\xi|$. The auxilliary problem reads

$$V(t,x) = \inf_{a \in \mathbb{A}_{[t,T]}} \left\{ \left| y_T^{t,x,a} \right| \bigvee \max_{s \in [t,T]} \left(1 - \left| y_s^{t,x,a} \right| \right) \right\}, \quad \begin{cases} (-\partial_t V + |\partial_{x_1} V|) \wedge (V - g) = 0, \\ V(T,x) = (|x_1| - x_2) \vee (1 - |x_1|). \end{cases}$$







Dynamical formulation ([ABZ13])

Dynamic programming principle For all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $h \in [0, T-t]$,

$$V(t,x) = \inf \left\{ V(t+h, y_{t+h}^{t,x,a}) \bigvee \max_{s \in [t,t+h]} g\left(y_s^{t,x,a}\right) \,\middle|\, a(\cdot) \in \mathbb{A}_{[t,t+h]} \right\}. \qquad \text{(DPP)}$$

Dynamical formulation ([ABZ13])

Dynamic programming principle For all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $h \in [0, T-t]$,

$$V(t,x) = \inf\left\{ V(t+h, y^{t,x,a}_{t+h}) \bigvee \max_{s\in[t,t+h]} g\left(y^{t,x,a}_s\right) \;\middle|\; a(\cdot) \in \mathbb{A}_{[t,t+h]} \right\}. \qquad \text{(DPP)}$$

Let $N \in \mathbb{N}$, $\Delta t = T/N$ and $t_n = n\Delta t$. Introduce a first discretization of (DPP) by

$$V^n(x) := \inf\left\{ V^{n+1}(F^a_{\Delta t}(x)) \bigvee G^a_{\Delta t}(x) \;\middle|\; a \in \mathsf{Mes}(\mathbb{R}^d, A) \right\}, \quad V^N(x) = \mathfrak{J}(x) \vee g(x),$$

where $F^a_{\Delta t}(x)$ is a consistant approximation of $y^{t,x,a}_{t+\Delta t}$, and $G^a_{\Delta t}(x)$ approximates $\max_{s\in[t,T]} g(y^{t,x,a}_s)$.

Dynamical formulation ([ABZ13])

Dynamic programming principle For all $(t,x) \in [0,T] \times \mathbb{R}^d$ and $h \in [0, T-t]$,

$$V(t,x) = \inf \left\{ V(t+h, y^{t,x,a}_{t+h}) \bigvee \max_{s\in[t,t+h]} g\left(y^{t,x,a}_s\right) \;\middle|\; a(\cdot) \in \mathbb{A}_{[t,t+h]} \right\}. \qquad \text{(DPP)}$$

Let $N \in \mathbb{N}$, $\Delta t = T/N$ and $t_n = n\Delta t$. Introduce a first discretization of (DPP) by

$$V^n(x) := \inf \left\{ V^{n+1}(F^a_{\Delta t}(x)) \bigvee G^a_{\Delta t}(x) \;\middle|\; a \in \mathsf{Mes}(\mathbb{R}^d, A) \right\}, \quad V^N(x) = \mathfrak{J}(x) \vee g(x),$$

where $F^a_{\Delta t}(x)$ is a consistant approximation of $y^{t,x,a}_{t+\Delta t}$, and $G^a_{\Delta t}(x)$ approximates $\max_{s\in[t,T]} g(y^{t,x,a}_s)$.

Under natural assumptions, $V^n(x) \to V(t_n, x)$ locally uniformly when $\Delta t \to 0$.

Table of Contents

Expression

Let $(t_n)_{n\in[\![0,N]\!]}$ be a discretization of $[0,T]$, and $\hat{\mathcal{A}}^n_\Theta \subset \text{Mes}\left(\mathbb{R}^d, A\right)$ be approximation spaces.

(A3) Let $\hat{y}^{t_n,x,a}_{t_{n+1}} = \hat{F}_n(x,a)$ be a consistant scheme s.t. $\hat{F}_n(\cdot,a)$ is bijective for small Δt.

Expression

Let $(t_n)_{n\in[\![0,N]\!]}$ be a discretization of $[0,T]$, and $\hat{\mathcal{A}}^n_\Theta \subset \text{Mes}\left(\mathbb{R}^d, A\right)$ be approximation spaces.

(A3) Let $\hat{y}^{t_n,x,a}_{t_{n+1}} = \hat{F}_n(x,a)$ be a consistant scheme s.t. $\hat{F}_n(\cdot,a)$ is bijective for small Δt.

Lagrangian scheme Let $(\mu^n)_{n\in[\![0,N-1]\!]} \subset \mathscr{P}_1(\mathbb{R}^d)$ be densities, and define

$$\begin{cases} \hat{V}^N := \mathfrak{J} \vee g, \qquad \hat{V}^n(x) := \hat{V}^{n+1}\left(\hat{y}^{t_n,x,\hat{a}^n}_{t_{n+1}}\right) \bigvee g\,(x) & \text{(1a)} \\ \text{where} \quad \hat{a}^n \in \underset{a\in\hat{\mathcal{A}}^n_\Theta}{\text{argmin}} \displaystyle\int_{x\in\mathbb{R}^d} \left[\hat{V}^{n+1}\left(\hat{y}^{t_n,x,a}_{t_{n+1}}\right) \bigvee g\,(x)\right] d\mu^n(x). & \text{(1b)} \end{cases}$$

Expression

Let $(t_n)_{n\in[\![0,N]\!]}$ be a discretization of $[0,T]$, and $\hat{\mathcal{A}}_\Theta^n \subset \mathrm{Mes}\left(\mathbb{R}^d, A\right)$ be approximation spaces.

(A3) Let $\hat{y}_{t_{n+1}}^{t_n,x,a} = \hat{F}_n(x,a)$ be a consistant scheme s.t. $\hat{F}_n(\cdot,a)$ is bijective for small Δt.

Lagrangian scheme Let $(\mu^n)_{n\in[\![0,N-1]\!]} \subset \mathscr{P}_1(\mathbb{R}^d)$ be densities, and define

$$\begin{cases} \hat{V}^N := \mathfrak{J} \vee g, \qquad \hat{V}^n(x) := \hat{V}^{n+1}\left(\hat{y}_{t_{n+1}}^{t_n,x,\hat{a}^n}\right) \bigvee g\left(x\right) & \text{(1a)} \\ \text{where} \quad \hat{a}^n \in \operatorname*{argmin}_{a\in\hat{\mathcal{A}}_\Theta^n} \int_{x\in\mathbb{R}^d} \left[\hat{V}^{n+1}\left(\hat{y}_{t_{n+1}}^{t_n,x,a}\right) \bigvee g\left(x\right)\right] d\mu^n(x). & \text{(1b)} \end{cases}$$

Remark – Storage The approximations $\hat{V}^n$ are just notations (only $(\hat{a}^n)_{n\in[\![0,N-1]\!]}$ is stored).

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.
- Introduction of a substep approximation of $\max_{s\in[t_n,t_{n+1}]} g(y_s^{t,x,a})$ (keeping a fixed):

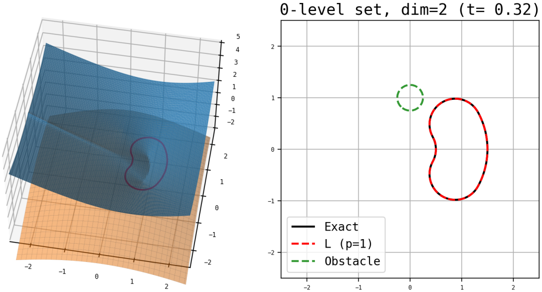


Figure: Without substeps

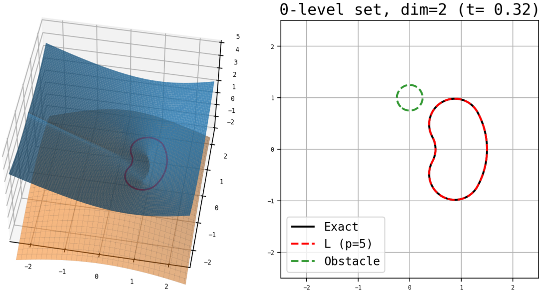


Figure: With substeps

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.
- Introduction of a substep approximation of $\max_{s\in[t_n,t_{n+1}]} g(y_s^{t,x,a})$ (keeping a fixed):

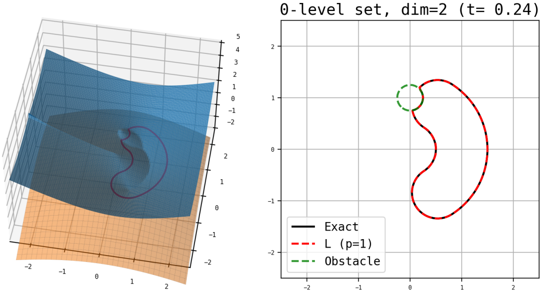


Figure: Without substeps

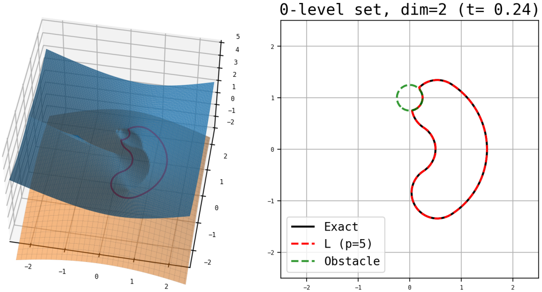


Figure: With substeps

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.
- Introduction of a substep approximation of $\max_{s\in[t_n,t_{n+1}]} g(y_s^{t,x,a})$ (keeping a fixed):

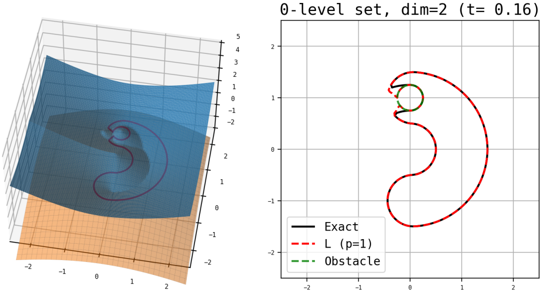


Figure: Without substeps

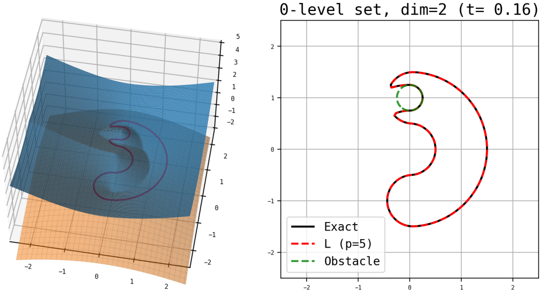


Figure: With substeps

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.
- Introduction of a substep approximation of $\max_{s\in[t_n,t_{n+1}]} g(y_s^{t,x,a})$ (keeping a fixed):

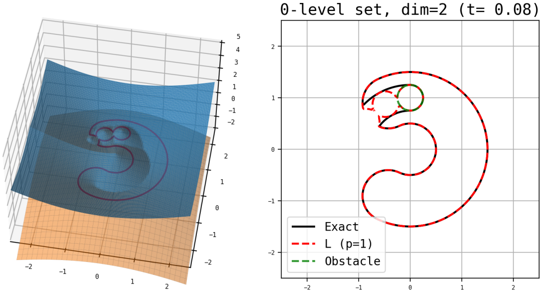


Figure: Without substeps

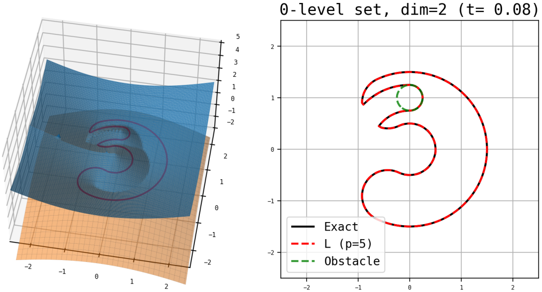


Figure: With substeps

Variants

- Storing $\hat{V}^n$: more memory, possible loss of precision, theoritical reduction of computation.
- Introduction of a substep approximation of $\max_{s \in [t_n, t_{n+1}]} g(y_s^{t,x,a})$ (keeping a fixed):

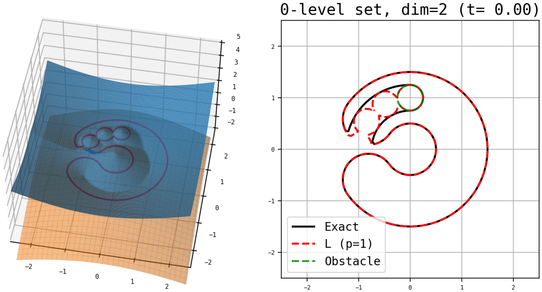


Figure: Without substeps

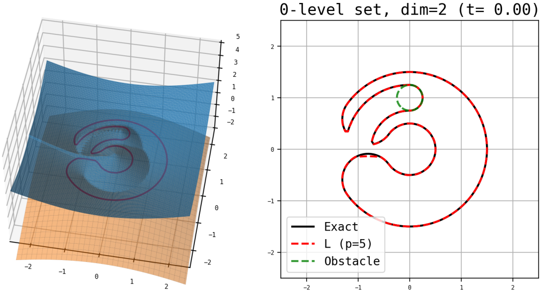


Figure: With substeps

Main result

(A4) Assume that

- $\hat{F}_n(x,\cdot)$ is continuous for small enough Δt, and $\left|\hat{F}_n(x,a)\right| \leqslant |x| + C\Delta t(1+|x|)$.
- The approximation spaces satisfy $\overline{\lim_{\Theta\to\infty} \hat{\mathcal{A}}^n_\Theta} = \mathrm{Lip}\left(\mathbb{R}^d, A\right)$ in $L^1_{\mu^n}$.
- The densities $\mu^n = \rho^n \mathcal{L}$ are such that $\hat{F}(\operatorname{supp} \rho^n) \subset \operatorname{supp} \rho^{n+1}$, and

$$C_{n,\Delta t} := \sup_{x\in\mathbb{R}^d} \sup_{a\in A} \frac{\rho^n(x)}{\rho^{n+1} \circ \hat{F}(x,a)} < \infty.$$

Main result

(A4) Assume that

- $\hat{F}_n(x,\cdot)$ is continuous for small enough Δt, and $\left|\hat{F}_n(x,a)\right| \leqslant |x| + C\Delta t(1+|x|)$.
- The approximation spaces satisfy $\overline{\lim_{\Theta\to\infty} \hat{\mathcal{A}}^n_\Theta} = \text{Lip}\left(\mathbb{R}^d, A\right)$ in $L^1_{\mu^n}$.
- The densities $\mu^n = \rho^n \mathcal{L}$ are such that $\hat{F}(\text{supp}\,\rho^n) \subset \text{supp}\,\rho^{n+1}$, and

$$C_{n,\Delta t} := \sup_{x\in\mathbb{R}^d} \sup_{a\in A} \frac{\rho^n(x)}{\rho^{n+1}\circ \hat{F}(x,a)} < \infty.$$

Convergence ([BPW22]) Under (A1) to (A4), $\lim_{\Theta\to\infty} \max_{n\in[\![0,N]\!]} \int |\hat{V}^n - V^n| d\mu^n = 0.$

Comments

- Statement in [BPW22] is a parametric result with explicit formula, where the approximation errors η_k may be chosen (at the expense of large Θ_n).

Comments

- Statement in [BPW22] is a parametric result with explicit formula, where the approximation errors η_k may be chosen (at the expense of large Θ_n).
- If analytical controls are Lipschitz and ρ^n are the uniform densities, result for arbitrary N.

Comments

- Statement in [BPW22] is a parametric result with explicit formula, where the approximation errors η_k may be chosen (at the expense of large Θ_n).
- If analytical controls are Lipschitz and ρ^n are the uniform densities, result for arbitrary N.

In the literature, this type of results is found in the neural network community. In particular,

- [HPBL21] and [BHLP22] analyze a similar problem in the context of stochastic optimization. The presented scheme is inspired from the *performance iteration* scheme of the authors, where the error analysis relies on diffusion, and related to the work of [BD07].

Comments

- Statement in [BPW22] is a parametric result with explicit formula, where the approximation errors η_k may be chosen (at the expense of large Θ_n).
- If analytical controls are Lipschitz and ρ^n are the uniform densities, result for arbitrary N.

In the literature, this type of results is found in the neural network community. In particular,

- [HPBL21] and [BHLP22] analyze a similar problem in the context of stochastic optimization. The presented scheme is inspired from the *performance iteration* scheme of the authors, where the error analysis relies on diffusion, and related to the work of [BD07].
- A similar error analysis is performed in [GPW20, GPW21], using GroupSort networks.

Comments

- Statement in [BPW22] is a parametric result with explicit formula, where the approximation errors η_k may be chosen (at the expense of large Θ_n).
- If analytical controls are Lipschitz and ρ^n are the uniform densities, result for arbitrary N.

In the literature, this type of results is found in the neural network community. In particular,

- [HPBL21] and [BHLP22] analyze a similar problem in the context of stochastic optimization. The presented scheme is inspired from the *performance iteration* scheme of the authors, where the error analysis relies on diffusion, and related to the work of [BD07].
- A similar error analysis is performed in [GPW20, GPW21], using GroupSort networks.
- Global regression is studied (for instance) in [SS18] (DGM), or [HL20] for BSDEs.

Table of Contents

Definition

Neural network Let L be a *number of layers*, and $(d_k)_{k\in[\![0,L]\!]}$ be natural numbers. A map $\mathscr{R} : \mathbb{R}^{d_0} \to \mathbb{R}^{d_L}$ is a *feedforward neural network* if it is of the form

$$\mathscr{R} = \sigma_L \circ \mathcal{L}_L \circ \cdots \circ \sigma_1 \circ \mathcal{L}_1, \qquad \sigma_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i} \text{ nonlinear}, \quad \mathcal{L}_i : \mathbb{R}^{d_{i-1}} \to \mathbb{R}^{d_i} \text{ linear.}$$

Definition

Neural network Let L be a *number of layers*, and $(d_k)_{k \in [\![0,L]\!]}$ be natural numbers. A map $\mathscr{R} : \mathbb{R}^{d_0} \to \mathbb{R}^{d_L}$ is a *feedforward neural network* if it is of the form

$$\mathscr{R} = \sigma_L \circ \mathcal{L}_L \circ \cdots \circ \sigma_1 \circ \mathcal{L}_1, \qquad \sigma_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i} \text{ nonlinear}, \quad \mathcal{L}_i : \mathbb{R}^{d_{i-1}} \to \mathbb{R}^{d_i} \text{ linear}.$$

In the sequel, $d_1 = \cdots = d_{L-1}$, $d_0 = d$ is the space dimension, and $d_L = \kappa$.

Definition

Neural network Let L be a *number of layers*, and $(d_k)_{k\in[\![0,L]\!]}$ be natural numbers. A map $\mathscr{R} : \mathbb{R}^{d_0} \to \mathbb{R}^{d_L}$ is a *feedforward neural network* if it is of the form

$$\mathscr{R} = \sigma_L \circ \mathcal{L}_L \circ \cdots \circ \sigma_1 \circ \mathcal{L}_1, \qquad \sigma_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i} \text{ nonlinear}, \quad \mathcal{L}_i : \mathbb{R}^{d_{i-1}} \to \mathbb{R}^{d_i} \text{ linear}.$$

In the sequel, $d_1 = \cdots = d_{L-1}$, $d_0 = d$ is the space dimension, and $d_L = \kappa$.

- Various activations (ReLU $\max.(0, x)$, sigmoid $(1 + e^{-x}_{.})^{-1}$, GroupSort $\text{sort}_{\downarrow}(x)$)...

Definition

Neural network Let L be a *number of layers*, and $(d_k)_{k\in[\![0,L]\!]}$ be natural numbers. A map $\mathscr{R} : \mathbb{R}^{d_0} \to \mathbb{R}^{d_L}$ is a *feedforward neural network* if it is of the form

$$\mathscr{R} = \sigma_L \circ \mathcal{L}_L \circ \cdots \circ \sigma_1 \circ \mathcal{L}_1, \qquad \sigma_i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i} \text{ nonlinear}, \quad \mathcal{L}_i : \mathbb{R}^{d_{i-1}} \to \mathbb{R}^{d_i} \text{ linear.}$$

In the sequel, $d_1 = \cdots = d_{L-1}$, $d_0 = d$ is the space dimension, and $d_L = \kappa$.

- Various activations (ReLU $\max.(0, x)$, sigmoid $(1 + e^{-x}_{.})^{-1}$, GroupSort $\text{sort}_{\downarrow}(x)$)...
- Density in the space of continuous functions under mild assumptions (Lemma 16.1 of [GKKW02]).

Definition

Neural network Let L be a *number of layers*, and $(d_k)_{k\in[\![0,L]\!]}$ be natural numbers. A map $\mathscr{R}:\mathbb{R}^{d_0}\to\mathbb{R}^{d_L}$ is a *feedforward neural network* if it is of the form

$$\mathscr{R}=\sigma_L\circ\mathcal{L}_L\circ\cdots\circ\sigma_1\circ\mathcal{L}_1,\qquad \sigma_i:\mathbb{R}^{d_i}\to\mathbb{R}^{d_i}\text{ nonlinear},\quad \mathcal{L}_i:\mathbb{R}^{d_{i-1}}\to\mathbb{R}^{d_i}\text{ linear.}$$

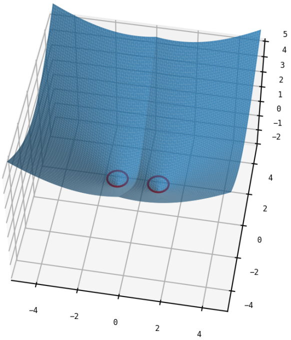
In the sequel, $d_1=\cdots=d_{L-1}$, $d_0=d$ is the space dimension, and $d_L=\kappa$.

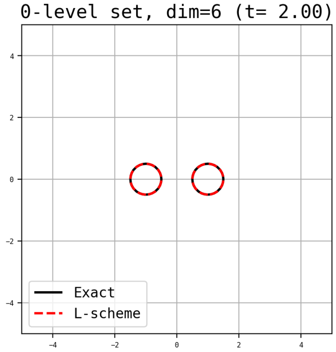
- Various activations (ReLU $\max.(0,x)$, sigmoid $(1+e^{-x}_{.})^{-1}$, GroupSort $\text{sort}_{\downarrow}(x)$)...
- Density in the space of continuous functions under mild assumptions (Lemma 16.1 of [GKKW02]).
- In practice, approximation very sensitive to the correct structure of the network.

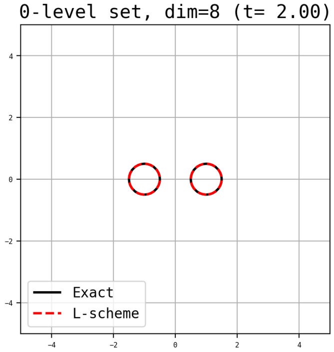
Eikonal equation (1/2)

We consider $T = 2$ and the obstacle-free Eikonal equation

$$-\partial_t V(t,x) + \max_{a \in \overline{\mathscr{B}}(0,1)} \langle \nabla V(t,x), a \rangle = 0, \quad \text{with} \quad V(T,x) = \min\left(|x + e_1|, |x - e_1|\right).$$



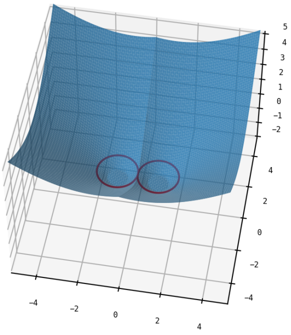


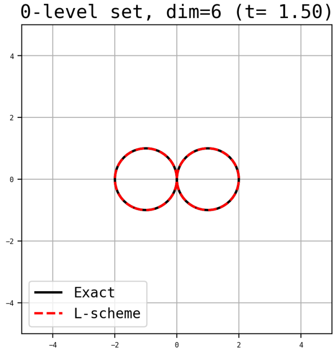


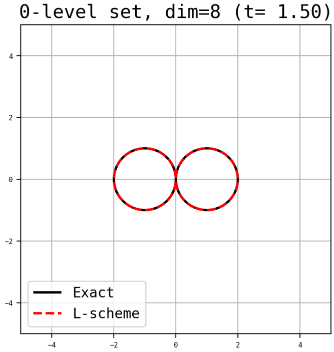
Eikonal equation (1/2)

We consider $T = 2$ and the obstacle-free Eikonal equation

$$-\partial_t V(t,x) + \max_{a \in \overline{\mathscr{B}}(0,1)} \langle \nabla V(t,x), a \rangle = 0, \quad \text{with} \quad V(T,x) = \min\left(|x+e_1|, |x-e_1|\right).$$



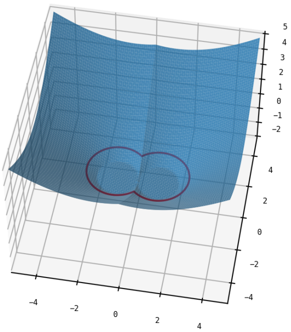


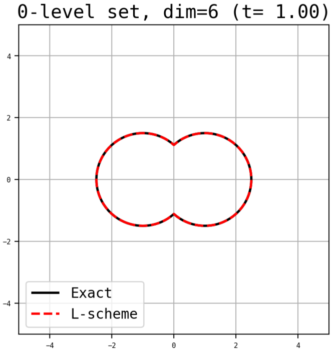


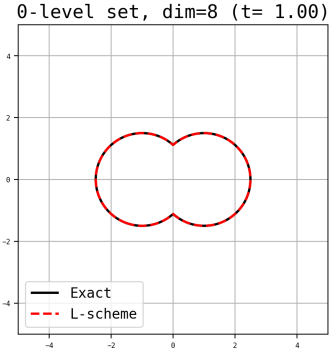
Eikonal equation (1/2)

We consider $T = 2$ and the obstacle-free Eikonal equation

$$-\partial_t V(t,x) + \max_{a \in \overline{\mathscr{B}}(0,1)} \langle \nabla V(t,x), a \rangle = 0, \quad \text{with} \quad V(T,x) = \min\left(|x+e_1|, |x-e_1|\right).$$



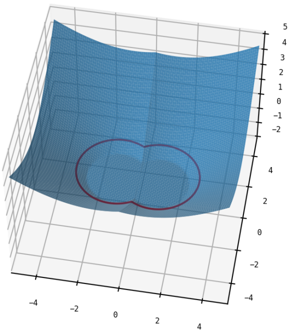


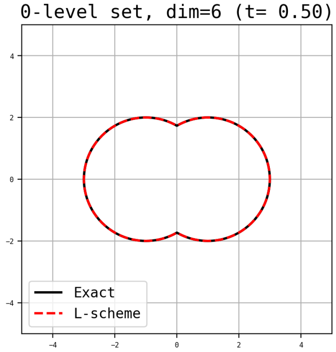


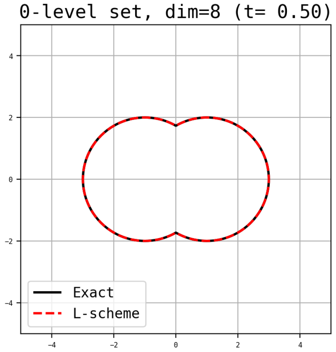
Eikonal equation (1/2)

We consider $T = 2$ and the obstacle-free Eikonal equation

$$-\partial_t V(t,x) + \max_{a \in \overline{\mathscr{B}}(0,1)} \langle \nabla V(t,x), a \rangle = 0, \quad \text{with} \quad V(T,x) = \min\left(|x+e_1|, |x-e_1|\right).$$



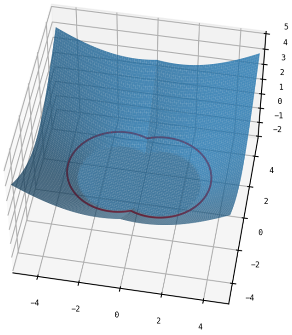


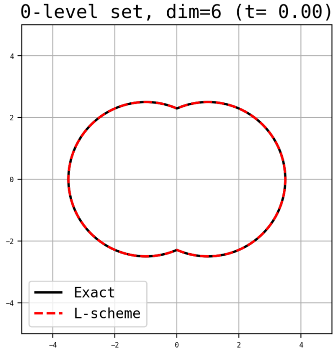


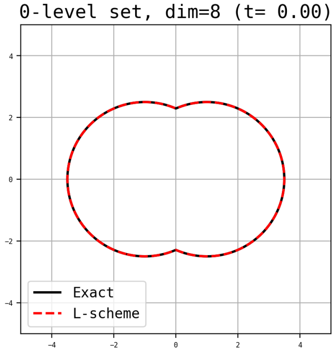
Eikonal equation (1/2)

We consider $T = 2$ and the obstacle-free Eikonal equation

$$-\partial_t V(t,x) + \max_{a \in \overline{\mathscr{B}}(0,1)} \langle \nabla V(t,x), a \rangle = 0, \quad \text{with} \quad V(T,x) = \min\left(|x+e_1|, |x-e_1|\right).$$



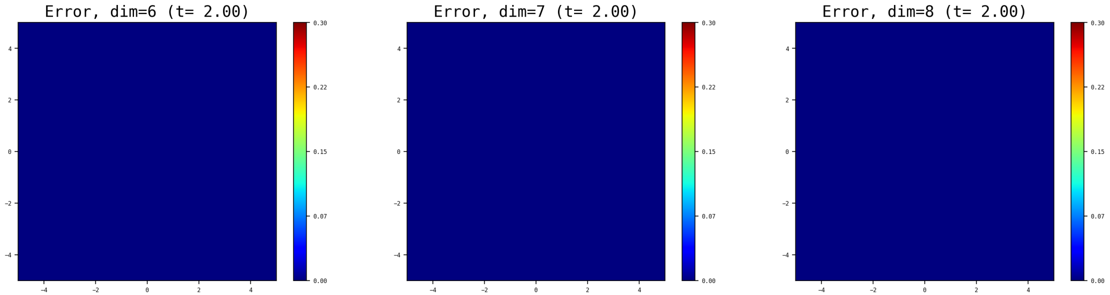




Eikonal equation (2/2)

Dim d	S.G it.	Global L_∞	Global L_1 rel.	Local L_∞	Local L_1 rel.	Time
6	100000	2.16e-02	1.96e-03	4.06e-04	1.58e-04	1h26
7	200000	5.00e-02	3.41e-03	1.51e-02	1.26e-04	3h55
8	400000	1.99e-01	1.81e-02	4.39e-04	2.19e-04	10h31

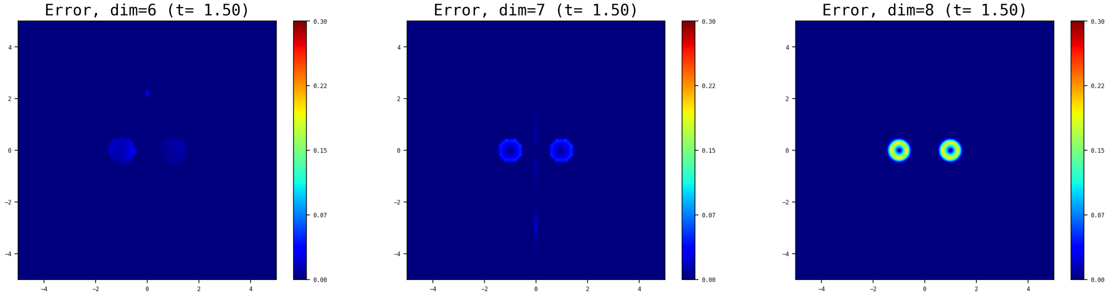
Table: Errors for the Eikonal equation, $N = 4$ iterations, 3 layers, 40 neurons



Eikonal equation (2/2)

Dim d	S.G it.	Global L_∞	Global L_1 rel.	Local L_∞	Local L_1 rel.	Time
6	100000	2.16e-02	1.96e-03	4.06e-04	1.58e-04	1h26
7	200000	5.00e-02	3.41e-03	1.51e-02	1.26e-04	3h55
8	400000	1.99e-01	1.81e-02	4.39e-04	2.19e-04	10h31

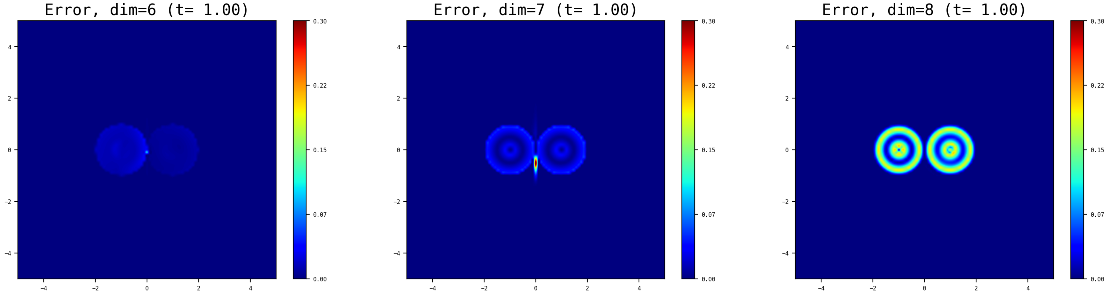
Table: Errors for the Eikonal equation, $N = 4$ iterations, 3 layers, 40 neurons



Eikonal equation (2/2)

Dim d	S.G it.	Global L_∞	Global L_1 rel.	Local L_∞	Local L_1 rel.	Time
6	100000	2.16e-02	1.96e-03	4.06e-04	1.58e-04	1h26
7	200000	5.00e-02	3.41e-03	1.51e-02	1.26e-04	3h55
8	400000	1.99e-01	1.81e-02	4.39e-04	2.19e-04	10h31

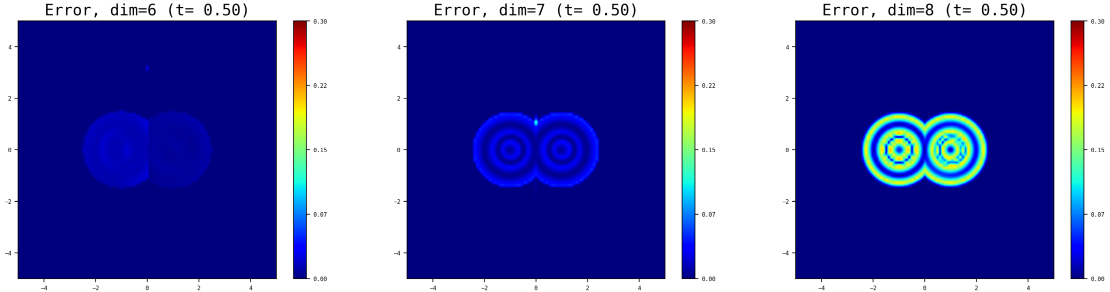
Table: Errors for the Eikonal equation, $N = 4$ iterations, 3 layers, 40 neurons



Eikonal equation (2/2)

Dim d	S.G it.	Global L_∞	Global L_1 rel.	Local L_∞	Local L_1 rel.	Time
6	100000	2.16e-02	1.96e-03	4.06e-04	1.58e-04	1h26
7	200000	5.00e-02	3.41e-03	1.51e-02	1.26e-04	3h55
8	400000	1.99e-01	1.81e-02	4.39e-04	2.19e-04	10h31

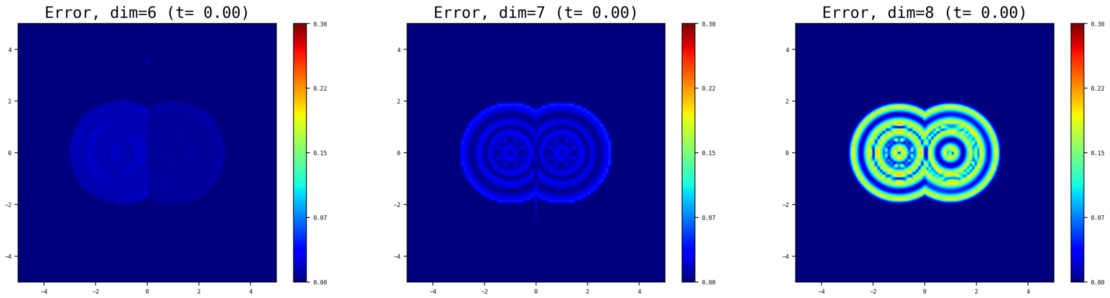
Table: Errors for the Eikonal equation, $N = 4$ iterations, 3 layers, 40 neurons



Eikonal equation (2/2)

Dim d	S.G it.	Global L_∞	Global L_1 rel.	Local L_∞	Local L_1 rel.	Time
6	100000	2.16e-02	1.96e-03	4.06e-04	1.58e-04	1h26
7	200000	5.00e-02	3.41e-03	1.51e-02	1.26e-04	3h55
8	400000	1.99e-01	1.81e-02	4.39e-04	2.19e-04	10h31

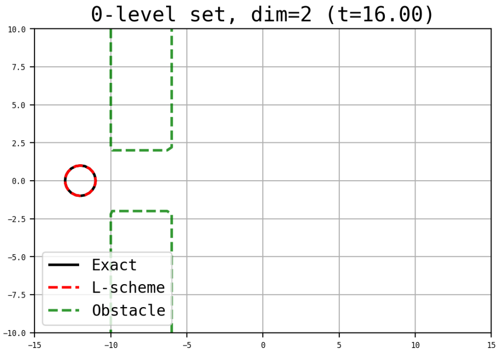
Table: Errors for the Eikonal equation, $N = 4$ iterations, 3 layers, 40 neurons

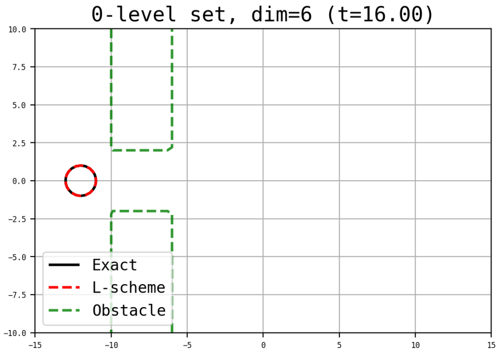


The door problem (1/2)

We consider the Eikonal-advection equation with $|b| > c > 0$:

$$\min\Big(- \partial_t V + \langle \nabla V, b\rangle + \max_{a\in \overline{\mathscr{B}}(0,1)} \langle \nabla V, ca\rangle\,, V - g\Big) = 0, \qquad V(T,\cdot) = \max\left(g, |\cdot| - 1\right).$$

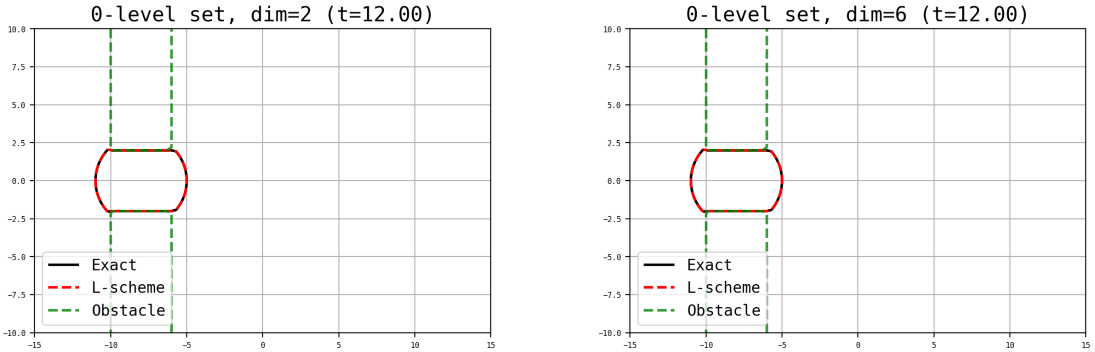




The door problem (1/2)

We consider the Eikonal-advection equation with $|b| > c > 0$:

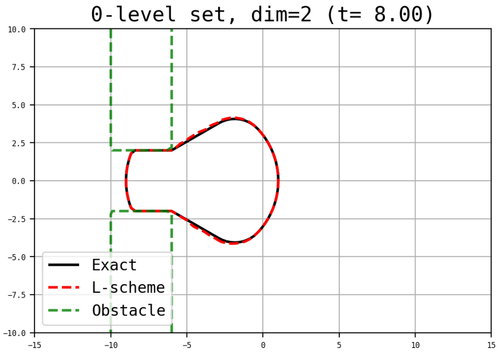
$$\min\left(-\partial_t V + \langle \nabla V, b\rangle + \max_{a\in\overline{\mathscr{B}}(0,1)} \langle \nabla V, ca\rangle, V - g\right) = 0, \qquad V(T,\cdot) = \max\left(g, |\cdot| - 1\right).$$

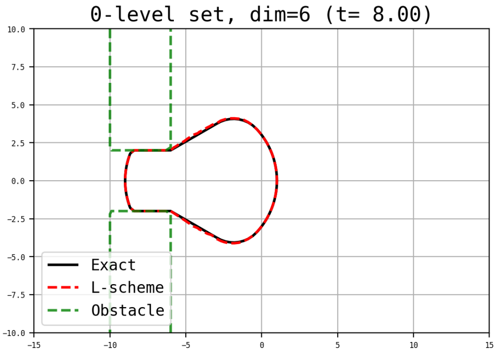


The door problem (1/2)

We consider the Eikonal-advection equation with $|b| > c > 0$:

$$\min\left(-\partial_t V + \langle \nabla V, b\rangle + \max_{a\in\overline{\mathscr{B}}(0,1)} \langle \nabla V, ca\rangle, V - g\right) = 0, \qquad V(T,\cdot) = \max\left(g, |\cdot| - 1\right).$$

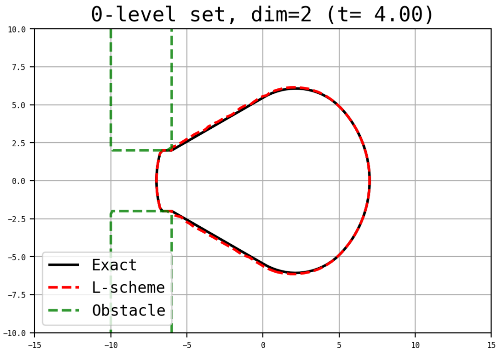


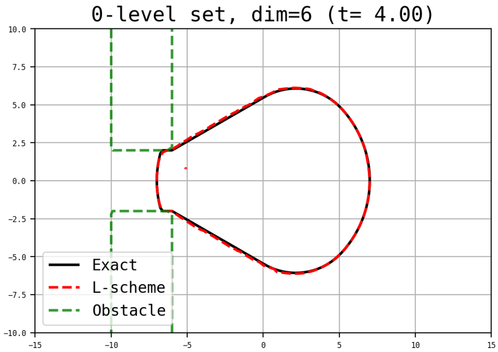


The door problem (1/2)

We consider the Eikonal-advection equation with $|b| > c > 0$:

$$\min\left(-\partial_t V + \langle \nabla V, b\rangle + \max_{a\in\overline{\mathscr{B}}(0,1)} \langle \nabla V, ca\rangle, V - g\right) = 0, \qquad V(T,\cdot) = \max\left(g, |\cdot| - 1\right).$$

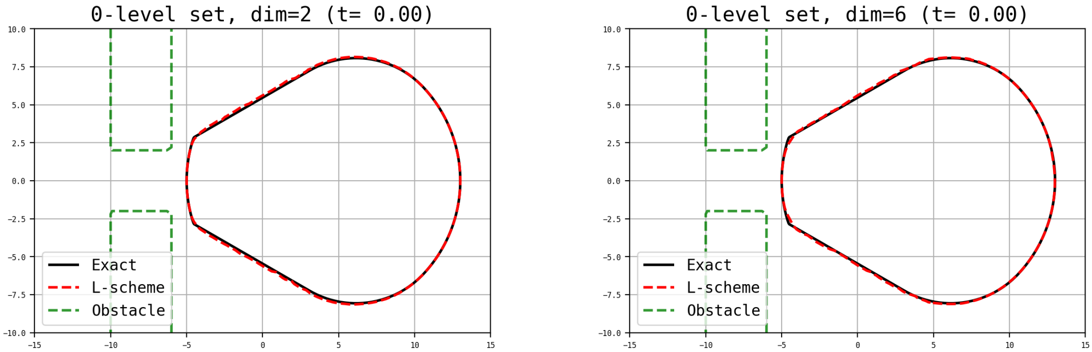




The door problem (1/2)

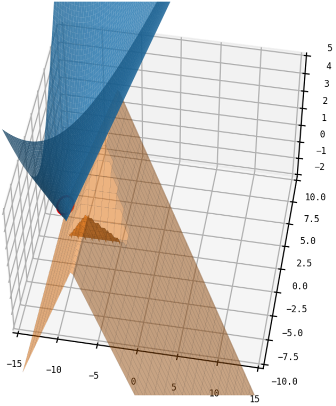
We consider the Eikonal-advection equation with $|b| > c > 0$:

$$\min\Big(-\partial_t V + \langle \nabla V, b\rangle + \max_{a\in\overline{\mathscr{B}}(0,1)} \langle \nabla V, ca\rangle\,, V - g\Big) = 0, \qquad V(T,\cdot) = \max\left(g, |\cdot| - 1\right).$$

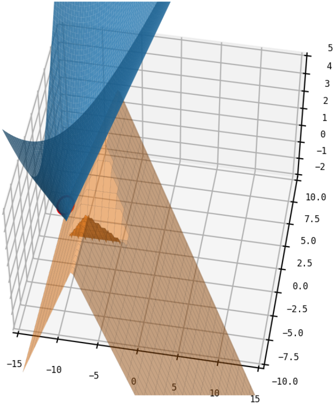


The door problem (2/2)

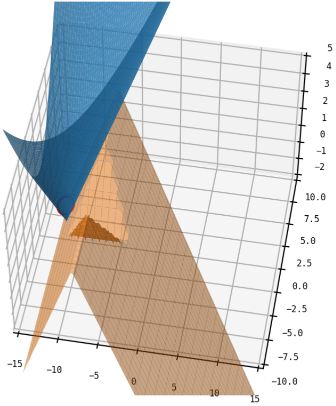
dim $d = 2$



dim $d = 6$

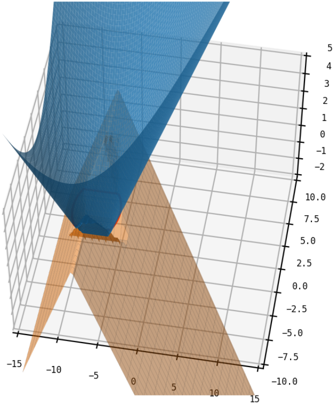


dim $d = 8$

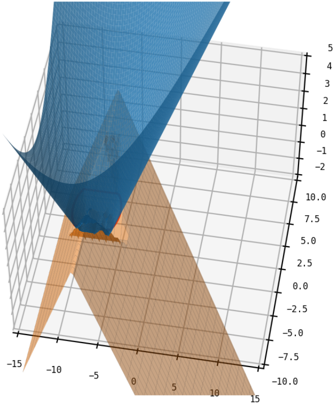


The door problem (2/2)

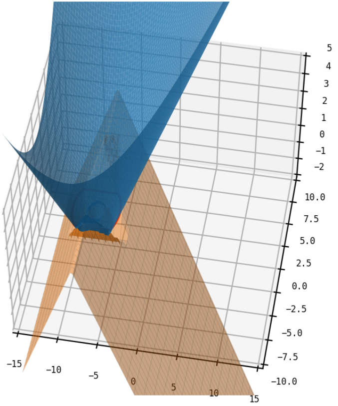
dim $d = 2$



dim $d = 6$

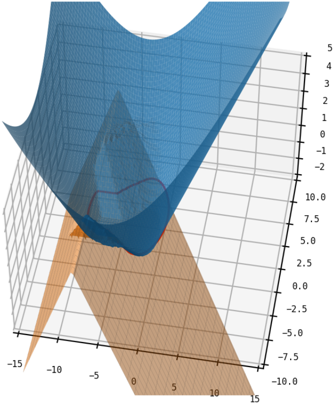


dim $d = 8$

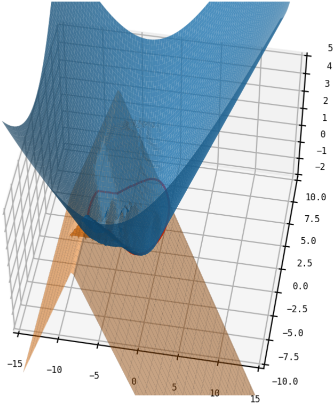


The door problem (2/2)

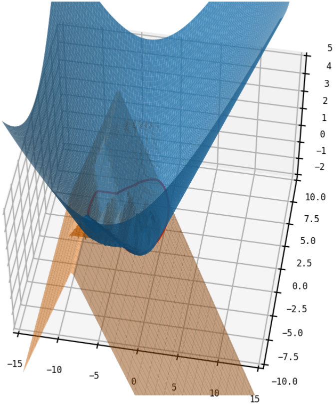
dim $d = 2$



dim $d = 6$

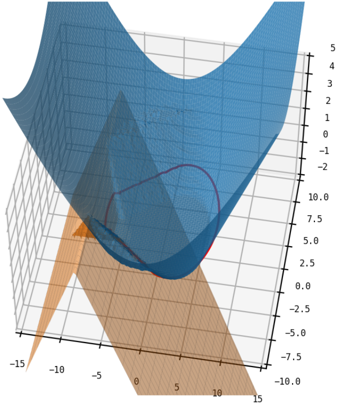


dim $d = 8$

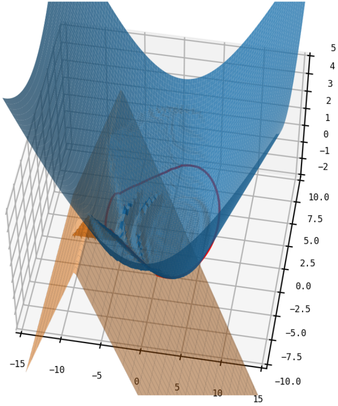


The door problem (2/2)

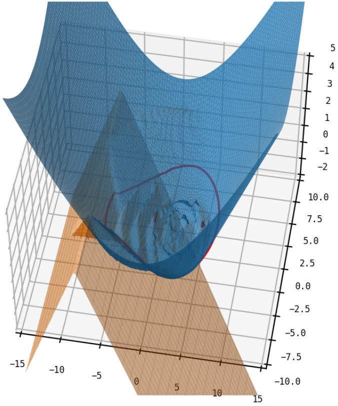
dim $d = 2$



dim $d = 6$

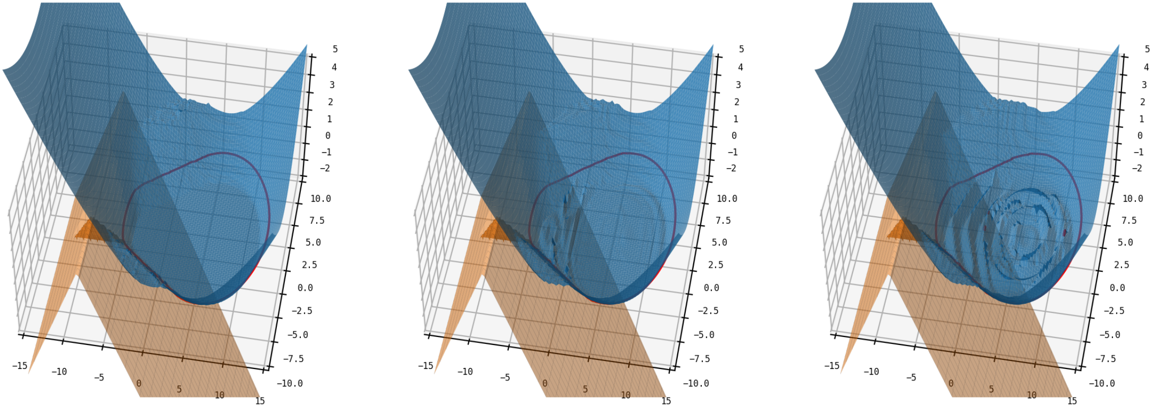


dim $d = 8$



The door problem (2/2)

dim $d = 2$ dim $d = 6$ dim $d = 8$



Thank you!

[ABZ13] Albert Altarovici, Olivier Bokanowski, and Hasnaa Zidani.
A general Hamilton-Jacobi framework for non-linear state-constrained control problems.
ESAIM: Control, Optimisation and Calculus of Variations, 19(2):337–357, April 2013.

[BD07] Christian Bender and Robert Denk.
A forward scheme for backward SDEs.
Stochastic Processes and their Applications, 117(12):1793–1812, December 2007.

[BHLP22] Achref Bachouch, Côme Huré, Nicolas Langrené, and Huyen Pham.
Deep neural networks algorithms for stochastic control problems on finite horizon: Numerical applications.
Methodology and Computing in Applied Probability, 24(1):143–178, March 2022.

[BPW22] Olivier Bokanowski, Averil Prost, and Xavier Warin.
Neural networks for first order HJB equations and application to front propagation with obstacle terms, October 2022.

[GKKW02] László Györfi, Michael Kohler, Adam Krzyżak, and Harro Walk.
A Distribution-Free Theory of Nonparametric Regression.
Springer Series in Statistics. Springer New York, New York, NY, 2002.

[GPW20] Maximilien Germain, Huyen Pham, and Xavier Warin.
Deep backward multistep schemes for nonlinear PDEs and approximation error analysis.
2020.

[GPW21] Maximilien Germain, Huyen Pham, and Xavier Warin.
Approximation error analysis of some deep backward schemes for nonlinear PDEs, September 2021.

[HL20] Jiequn Han and Jihao Long.
Convergence of the deep BSDE method for coupled FBSDEs.
Probability, Uncertainty and Quantitative Risk, 5(1):5, July 2020.

[HPBL21] Côme Huré, Huyên Pham, Achref Bachouch, and Nicolas Langrené.
Deep neural networks algorithms for stochastic control problems on finite horizon: Convergence analysis.
SIAM Journal on Numerical Analysis, 59(1):525–557, January 2021.

[SS18] Justin Sirignano and Konstantinos Spiliopoulos.
DGM: A deep learning algorithm for solving partial differential equations.
Journal of Computational Physics, 375:1339–1364, December 2018.