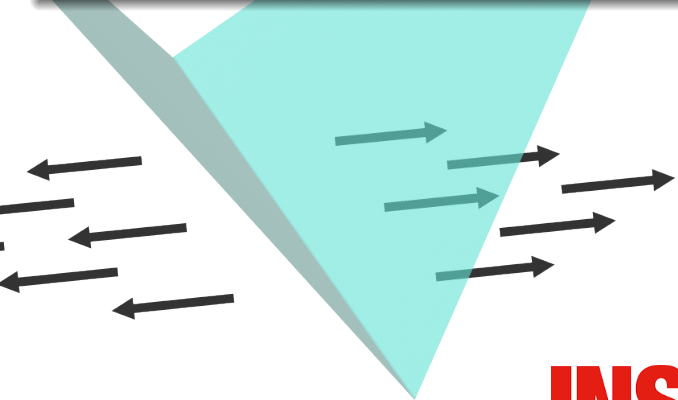
Monge-Ampère

A newbie's understanding of the story of this equation



Averil Aussedat

Séminaire LMI/LMRS

10 décembre 2024

INSA LMI - EA 3226 anr

Monge-Ampère

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Def The Monge-Ampère (MA) equation looks for a convex function $u : \Omega \to \mathbb{R}$ satisfying

(MA) $$\det\left(\nabla^2 u(x)\right) = \frac{f(x)}{g(\nabla u(x))},$$ and boundary conditions.

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In general, the right hand-side can be written $\overline{f}(x, u(x), \nabla u(x))$, but let us stick to (MA).

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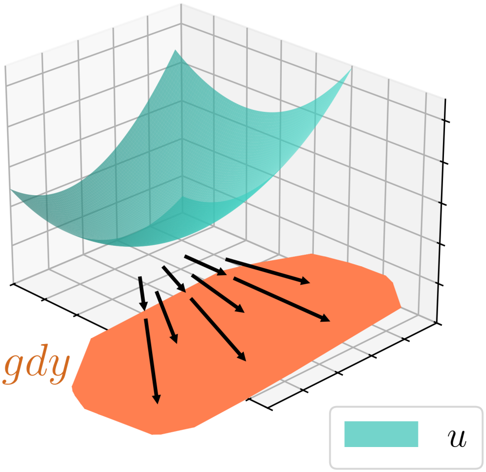

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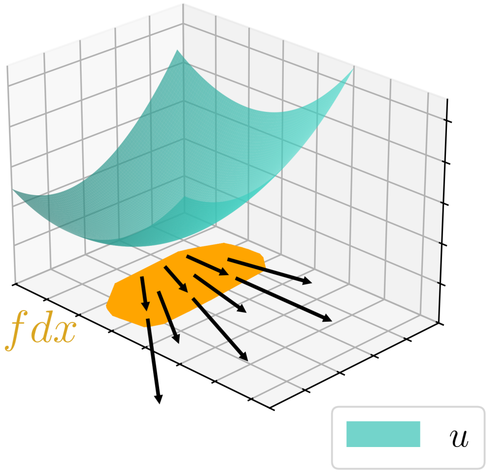

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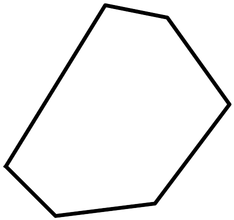

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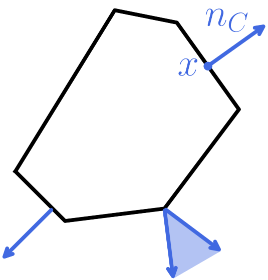

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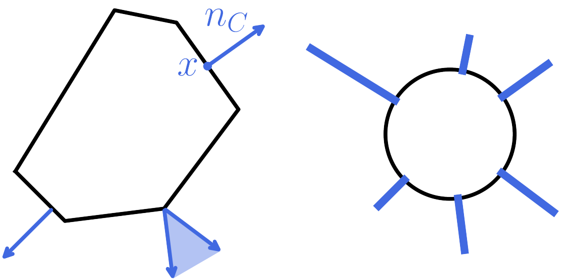

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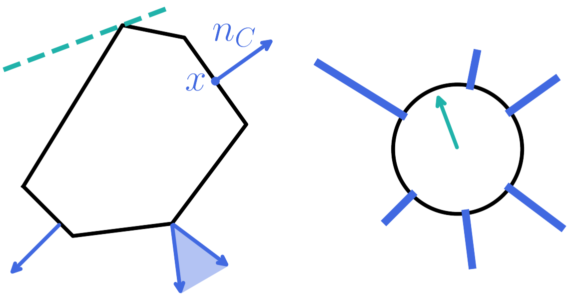

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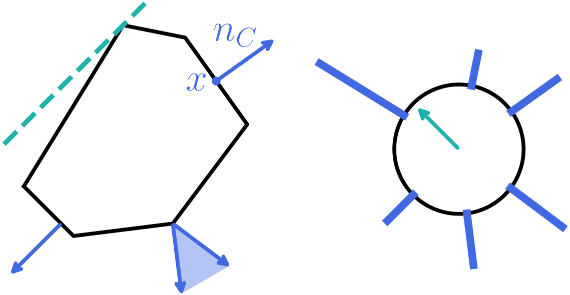

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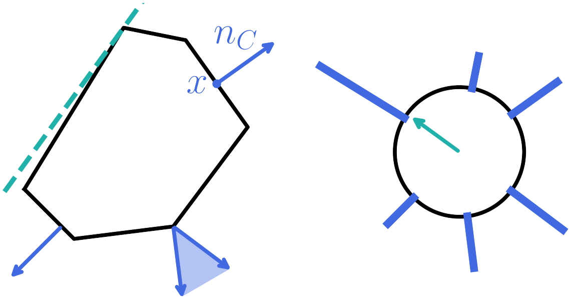

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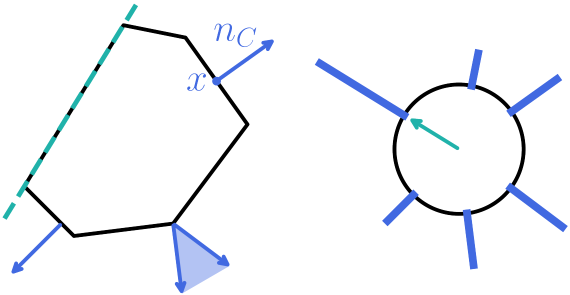

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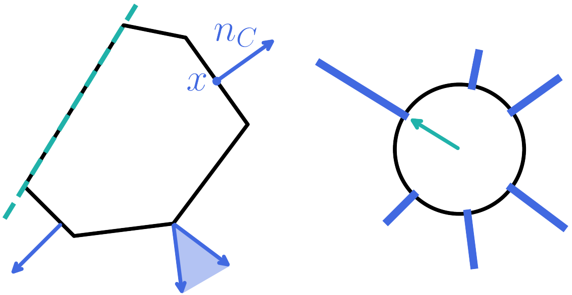

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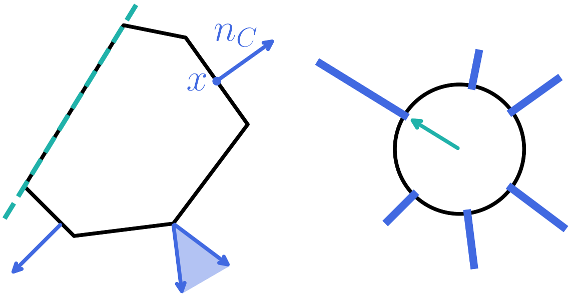
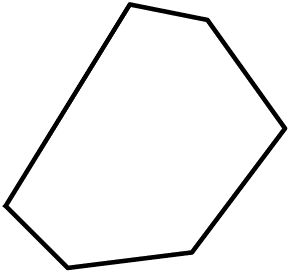


Minkowski's problem Given σ a measure on $\mathbb{S}^{d-1}$, find a convex C satisfying (1).

Minkowski's solution (1/2)



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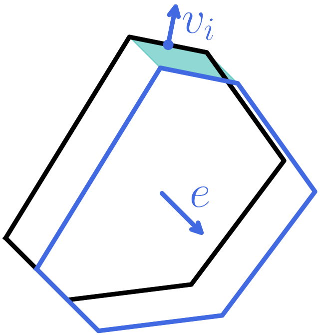

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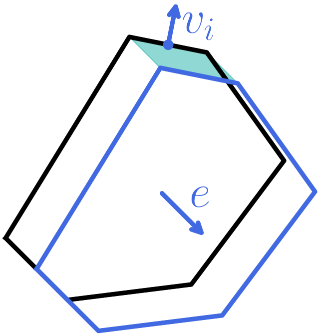

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Theorem – Minkowski [Min97] Any measure $\sigma = \sum_{i=1}^{N} \sigma_i \delta_{v_i}$ satisfying (2) is the surface area of a polyhedron that is unique up to translations.

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By Alexandrov's mapping lemma, Im is onto[1].

[1] A. V. Pogorelov. *The Minkowski Multidimensional Problem*. Scripta Series in Mathematics. Washington : New York, 1978.

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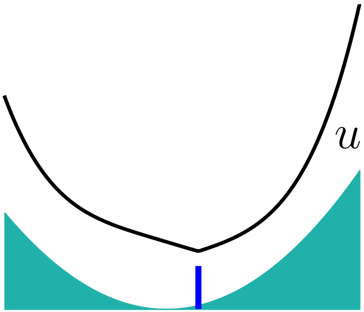

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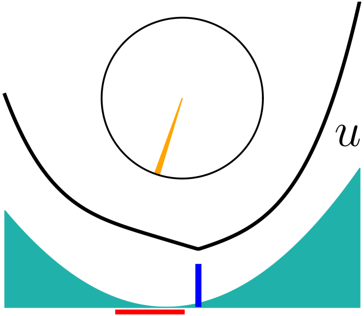

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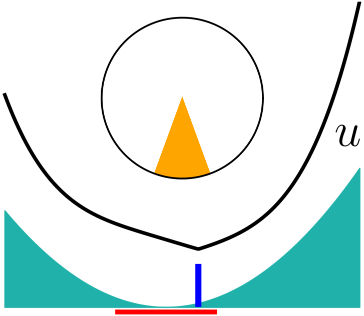

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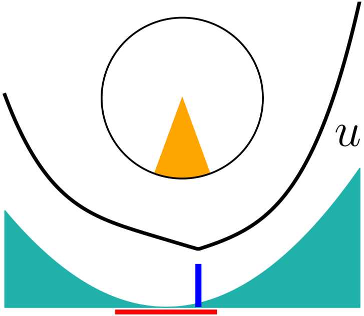

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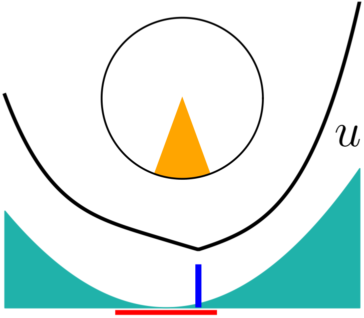

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If $u \in \mathcal{C}^2$ and convex, the change of variable $v = \nabla u(x)$ gives $dv = \det \nabla^2 u(x) dx$, and

$$\mu_u(A) = \mathrm{Leb}\left(\bigcup_{x\in A}\{\nabla u(x)\}\right) = \int_{v\in\mathbb{R}^d} \mathbb{1}_{\nabla u(A)}(v)dv = \int_{x\in A} \det \nabla^2 u(x)dx.$$

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- **existence:** by Perron's method. Uniqueness by comparison, hence *full well-posedness!*

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... but all more or less equivalent[2], and sharing estimates and tools.

[2] R. R. Jensen. Uniformly Elliptic PDEs with Bounded, Measurable Coefficients. *Journal of Fourier Analysis and Applications*, 2(3):237–259, June 1995

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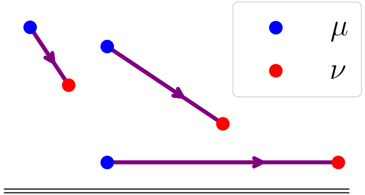

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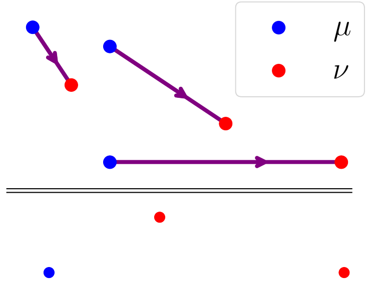

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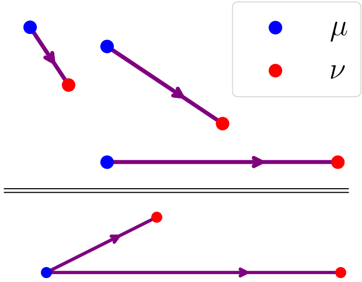


The ~~Monge~~ **Kantorovich** in Monge-Ampère

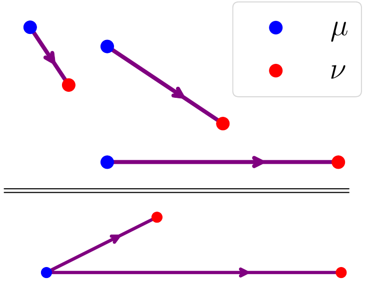
(MA) $$g(\nabla u(x)) \det \nabla^2 u(x) = f(x).$$

Introduce $p : x \mapsto \nabla u(x)$. As we saw, (MA) imposes that the measure $\mu := f dx$ is pushed on $\nu := g dy$ by $y = p(x)$. In the notations of optimal transport, $p\#\mu = \nu$.

Def – MK problem Given μ, ν measures, find ~~a map p~~ **a measure η** minimizing

$$\cancel{\int_{x\in\Omega} |p(x)|^2 d\mu(x)} \quad \boldsymbol{\int_{(x,y)\in\Omega^2} |y-x|^2 \, d\eta(x)}$$

among the ~~maps such that $p\#\mu = \nu$~~ **plans such that $\pi_x\#\eta = \mu$ and $\pi_y\#\eta = \nu$.**



Wonders of Kantorovich relaxation

Theorem – Existence of an optimal plan Let μ, ν be Borel probability measures such that $\int |x|^2 \, d\mu < \infty$ and $\int |x|^2 \, d\nu < \infty$. Then there exists an optimal plan η.

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Theorem – Brenier-McCann theorem Assume that $\mu = f dx$ has a density with respect to the Lebesgue measure. Then

- the optimal plan η is unique,
- it has the form $\eta = (id, p)\#\mu$ for some vector field $p \in L^2_\mu$,
- there exists a convex function $u : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $p(x) \in \partial u(x)$ for $\mu-$almost every x.

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The minimal condition for this theorem to hold has been found by Gigli [Gig11].

Brenier solutions (the last ones, I promise)

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Def – Brenier solution Let $\mu = f(\cdot)dx$. A lower semi-continuous convex function $u : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a Brenier solution of (MA) if $\nabla u \# \mu = \nu$.

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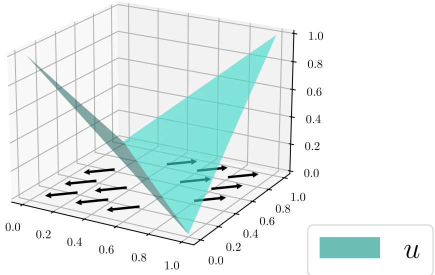
- Stability?
- Regularity?
- Links with Alexandrov/viscosity/... solutions?



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A (cheating) stability result using [Vil09, Theorem 5.20]:

Proposition If $(\mu_n)_n \rightharpoonup_n \mu$, $(\nu_n)_n \rightharpoonup_n \nu$, and there exists a unique optimal transport map T between μ and ν, then any family $(T_n)_n$ of optimal transport maps between μ_n and ν_n converge to T.

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Not very interesting compared to the "true" stability result holding at the level of plans. Some quantitative stability if $\mu = f\,dx$ is sufficiently regular: for instance, in [MDC20], the

Proposition If $0 < c_f \leqslant f \leqslant C_f < \infty$ uniformly over a compact convex set $\mathcal{X}$, then for any compact $K \subset \mathbb{R}^d$, there exists $C = C_{d,K,\mathcal{X}}$ such that

$$\|T_\nu - T_\omega\|_\mu \leqslant C d_{\mathcal{W},1}^{1/6}(\mu, \nu) \qquad \forall \mu, \nu \in \mathscr{P}_2(\Omega) \quad \text{with support in } K.$$

Some results on regularity

- Regularity results by Alexandrov and Pogorelov [Fig17, Theorem 2.22]: if $C \subset \mathbb{R}^d$ is convex, and σ_C has a bounded density with respect to $\mathcal{H}_{\mathbb{S}^{d-1}}$, then ∂C is of class $\mathcal{C}^1$.

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... and the story keeps writing itself.

Thank you !

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