# Follow the distance

## Viscosity solutions of monotone PDEs in some metric spaces


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# Classical viscosity solutions

Let $\Omega$ be an open set. Consider the equation (HJ)

$$\begin{aligned} H\left(x, u(x), D_x u(x), D^2_x u(x)\right) &= 0 \qquad x \in \Omega, \\ u(x) &= \mathfrak{J}(x) \qquad x \in \partial\Omega. \end{aligned}$$

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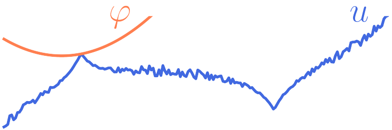


**Def** A function $u \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ is a viscosity solution of (HJ) if $u(x) = \mathfrak{J}(x)$, and

- for any $\varphi \in \mathcal{C}^2$ and any $x \in \Omega$ such that $u(x) = \varphi(x)$ and $u(y) \leqslant \varphi(y)$ around $x$,

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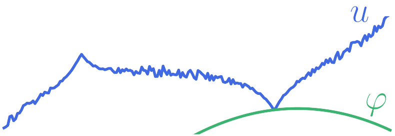


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# Monotonicity

Consider $H = H(x, r, p, X)$, with $x \in \mathbb{R}^d$, $r \in \mathbb{R}$, $p \in \mathsf{T}_x\mathbb{R}^d$ and $X \in \mathbb{M}_{d,d}$ symmetric.

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- Non-increasing monotonicity: for all $x, r, p$ and $X, Y$ such that $X \leqslant Y$ as matrices[1],

$$H(x, r, p, X) \geqslant H(x, r, p, Y).$$

[1] In the sense that $\langle Xv, v\rangle \leqslant \langle Yv, v\rangle$ for all vector $v$.

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- Increasing monotonicity: there exists $\gamma > 0$ such that either

$$H(x, r, p, X) - H(x, s, p, X) \geqslant \gamma(r - s),$$
$$H(x, r, p, X) - H(x, r, q, X) \geqslant \gamma \langle v, p - q \rangle, \text{ for some fixed } v, \qquad \cdots$$

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The first is (almost) *necessary for existence*; the second is (sometimes) *sufficient for uniqueness*.

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# Examples

- Canonical examples:

$$H(r, X) = r - \mathsf{Trace}(X), \qquad H(x, p) = |p| - n(x), \qquad H(r, X) = r - \det(X)$$

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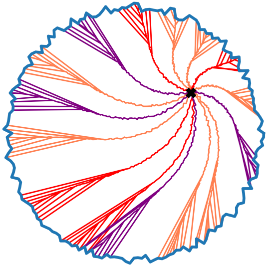
- Equations encoding monotone aggregation of information along characteristics:

$$H(x, r, p, X) = \sup_{a \in A} \inf_{b \in B} - \langle p, f[x, a, b] \rangle \, .$$

Extensions to second order, stochastic control with expectations.



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# The objective

Our aim is to study well-posedness of the parabolic equation

$$\begin{aligned} -\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) &= 0 && (t,x) \in [0,T) \times \Omega, \\ u(T,x) &= \mathfrak{I}(x) && x \in \Omega. \end{aligned}$$

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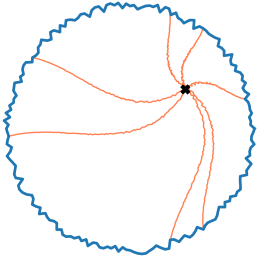
In the particular case where $H(x,p) = \sup_{v \in f[x]} -p(v)$, it is expected to characterize

$$V(t,x) := \inf_{\gamma \in \mathcal{S}_T^{t,x}} \mathfrak{J}(\gamma_T),$$

where $\mathcal{S}_T^{t,x} \subset \mathrm{AC}([t,T];\Omega)$ is the set of solutions of $\dot{\gamma}_s \in f[\gamma_s]$ issued from $x$ at time $t$.



# Setting

Consider $(\Omega, d)$ a complete geodesic metric space with a given curvature in the sense of Alexandrov: either

- a CAT(0) space, with $d^2(x,\gamma_t) \leqslant (1-t)d^2(x,\gamma_0) + td^2(x,\gamma_1) - t(1-t)d^2(\gamma_0,\gamma_1)$,

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- a CBB(0) space, with $d^2(x,\gamma_t) \geqslant (1-t)d^2(x,\gamma_0) + td^2(x,\gamma_1) - t(1-t)d^2(\gamma_0,\gamma_1)$,

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The squared distance is directionally differentiable along geodesics.

# Adapting each component of the definition

- **First-order calculus**

Taking a suitable closure of reparametrized geodesics gives a tangent cone $(\mathsf{T}_x\Omega, d_x(\cdot,\cdot))$ at $x$.
Let $\mathbb{T}$ be the set of $(x,p)$ with $p$ continuous and positively homogeneous from $\mathsf{T}_x\Omega$ to $\mathbb{R}$, and

$H : \mathbb{T} \to \mathbb{R}$. For instance $H(x,p) = \sup_{v \in f[x]} -p(v)$.

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- **Test functions**

$$\text{Let } \mathscr{T}_\pm := \left\{ (t,x) \mapsto \psi(t) \pm \sum_{n\in\mathbb{N}} \alpha_n d^2(\cdot, x_n) \;\middle|\; \begin{array}{l} \psi \in \mathcal{C}^1([0,T];\mathbb{R}), (\alpha_n)_{n\in\mathbb{N}} \in \ell^1, \\ \alpha_n \geqslant 0,\ (x_n)_{n\in\mathbb{N}} \text{ bounded in } \Omega. \end{array} \right\}$$

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- **Regularity** A function $u : \Omega \to \mathbb{R}$ is said locally uniformly upper semicontinuous (luusc) if $B \mapsto \sup_{x\in B} u(x)$ is usc over nonempty bounded sets endowed with the Hausdorff distance.

# Definition of viscosity solutions

Let $H : \mathbb{T} \to \mathbb{R}$, and consider

$$\begin{aligned} -\partial_t u(t,x) + H\left(x, D_x u(t,x)\right) &= 0 && (t,x) \in [0,T) \times \Omega, \\ u(T,x) &= \mathfrak{I}(x) && x \in \Omega. \end{aligned}$$

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**Def 1** A function $u \in \mathcal{C}(\overline{\Omega}; \mathbb{R})$ is a viscosity solution of (HJ) if $u(x) = \mathfrak{J}(x)$,

- it is luusc, and for any $(x, \varphi) \in \Omega \times \mathscr{T}_+$ such that $u(x) = \varphi(x)$ and $u(y) \leqslant \varphi(y)$,

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# A comparison principle

There exist $C_1, C_2 \geqslant 0$ such that for all $x, y \in \Omega$, $p, q \in \mathsf{T}_x\Omega$ and $a \geqslant 0$,

**(A1)**
$$|H(x,p) - H(x,q)| \leqslant C_1 \sup_{v \in T_x\Omega, |v|_x = 1} |p(v) - q(v)|,$$
$$H(y, -aD_y d^2(x,\cdot)) - H\left(x, aD_x d^2(\cdot, y)\right) \leqslant C_2\, d(x,y)\,(1 + ad(x,y)).$$

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**Theorem 1** Assume (A1). Let $u : \Omega \to \mathbb{R}$ be a bounded function satisfying (subsol), and $v : \Omega \to \mathbb{R}$ bounded satisfy (supersol). Then

$$\sup_{(t,x)\in[0,T]\times\Omega} u(t,x) - v(t,x) \leqslant \sup_{x\in\Omega} u(T,x) - v(T,x).$$

Arguments: doubling of variable, a smooth Ekeland principle, locally uniform semicontinuity.

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# CAT(0) spaces

The previous definition was introduced in CAT(0) spaces by [Jer22, JZ23]. In these spaces,

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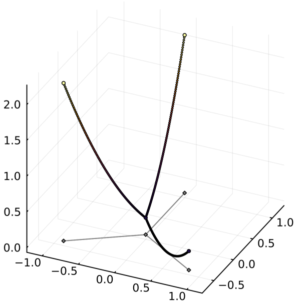

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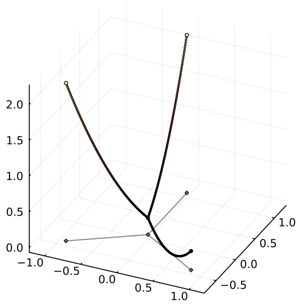


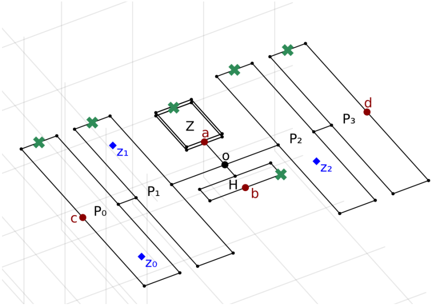

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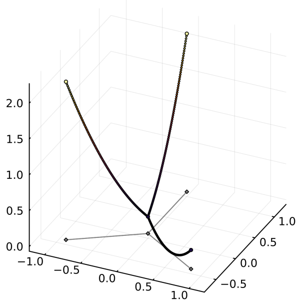


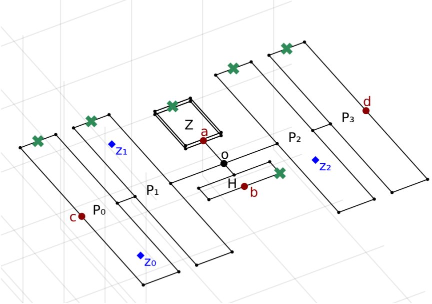


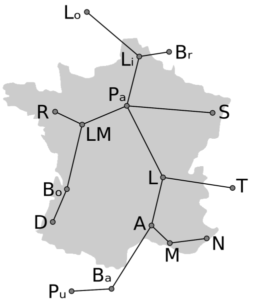

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$$\lim_{h \searrow 0} \frac{d(y_{t+h}, \Phi_{f(y_t, u(t))}(h, y_t))}{h} = 0.$$

# Current investigation (2/2)

**Existence of an optimal control**
Reformulation by EVIs allows extraction in $L^1(0,T;$ Banach space of energies$)$ and limit.

**Theorem** Let $f : \Omega \rightrightarrows \mathcal{C}(\Omega; \mathbb{R})$ be Lipschitz with compact values in a subset of Lipschitz and convex potentials. The closure of the set of solutions of $\mathring{y}_t \in f(y_t)$ issued from $x \in \Omega$ in AC$([0,T];\Omega)$ is given by the trajectories of $x \mapsto \overline{\mathrm{conv}}\, f(x)$.

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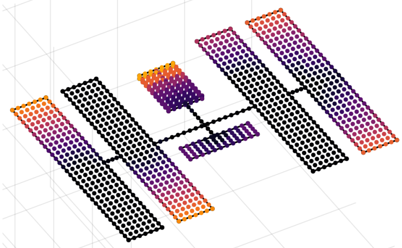
Assume $f$ and $\mathfrak{J}$ are Lipschitz. The value function of

$$\text{Minimize } \mathfrak{J}(y_T^{t,x,u}) \text{ over } u(\cdot) \in L^1(t,T;U)$$
$$\text{where } \mathring{y}_s = f(y_s, u(s)) \text{ and } y_t = x$$

is the unique viscosity solution of (HJB) with Hamiltonian $H(x,p) = \sup_{u \in U} -p(\nabla_x f(x,u))$.



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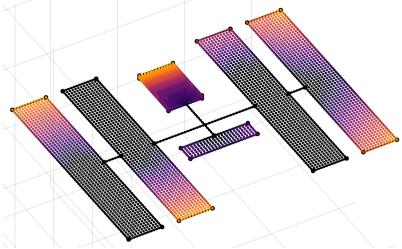

# Current investigation (2/2)

## Existence of an optimal control

Reformulation by EVIs allows extraction in $L^1(0,T;$ Banach space of energies$)$ and limit.

> **Theorem** Let $f : \Omega \rightrightarrows \mathcal{C}(\Omega;\mathbb{R})$ be Lipschitz with compact values in a subset of Lipschitz and convex potentials. The closure of the set of solutions of $\mathring{y}_t \in f(y_t)$ issued from $x \in \Omega$ in AC$([0,T];\Omega)$ is given by the trajectories of $x \mapsto \overline{\mathrm{conv}}\, f(x)$.

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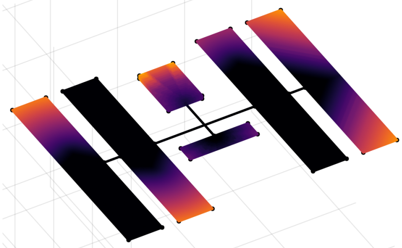

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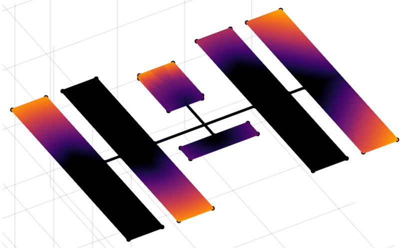

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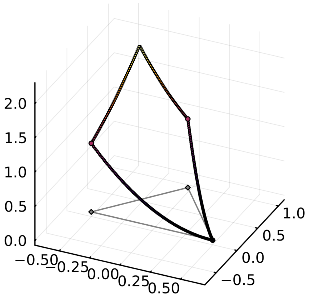

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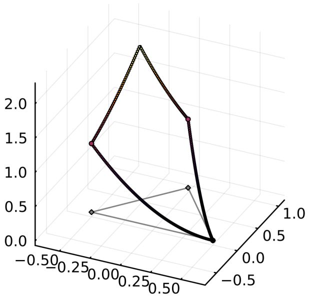


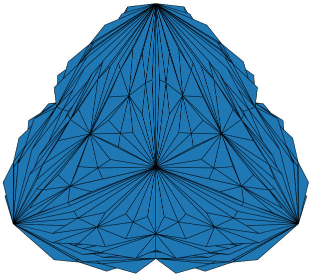

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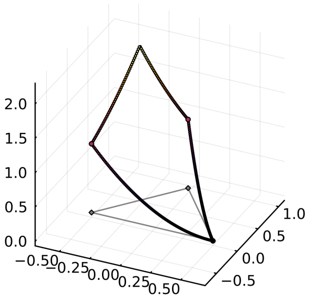

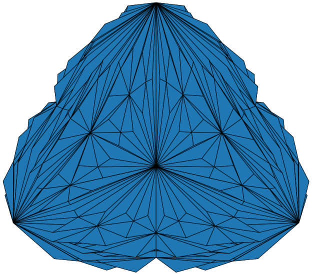
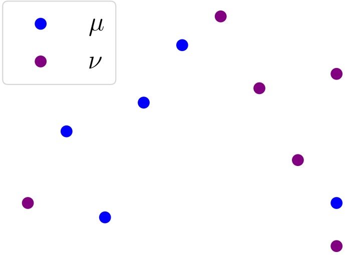

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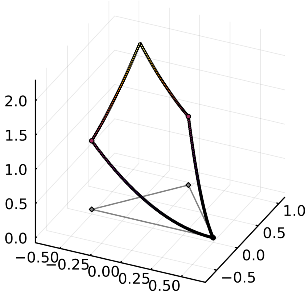


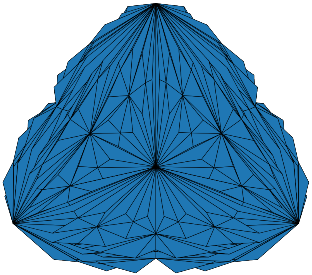

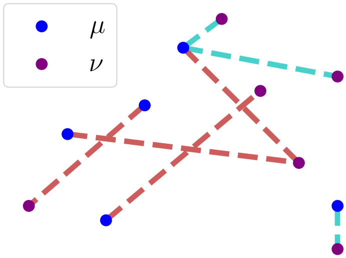

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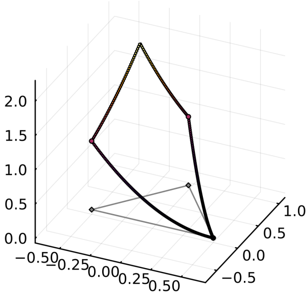


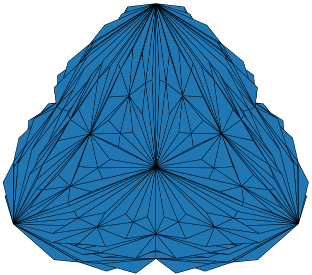

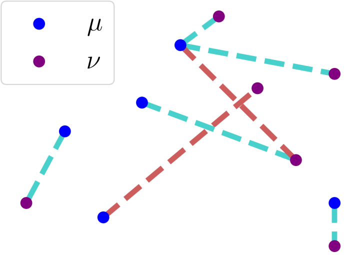

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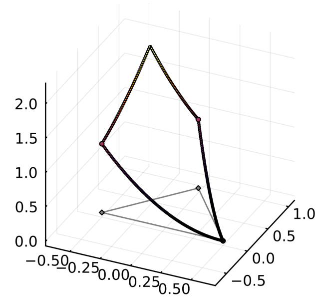


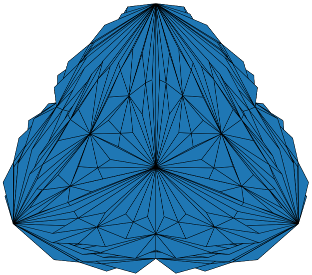

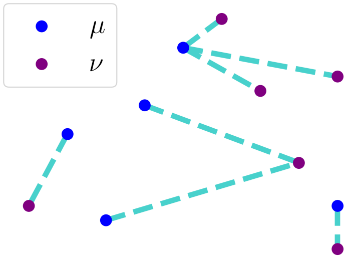

# Results in the Wasserstein space

Consider the control problem

$$\begin{aligned}&\text{Minimize} \quad \mathfrak{J}(\mu_T^{t,\nu,u}) \quad \text{over } u(\cdot) \in L^1(t,T;U),\\ &\text{where } \partial_s \mu_s + \text{div}\,(f[\mu_s, u(s)]\#\mu_s) = 0 \text{ for } s \in (t,T), \quad \mu_t = \nu.\end{aligned}$$

Well-posedness results for continuity inclusions in [BF21, BF23].

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**Theorem – Characterization [AJZ24]** Assume $f$ to be locally Lipschitz with linear growth, have convex $f[\mu, U]$, and $\mathfrak{J} : \mathscr{P}_2(\Omega) \to \mathbb{R}$ to be Lipschitz. Then

$$V(t,\nu) := \inf_{u(\cdot)\in L^1(t,T;U)} \mathfrak{J}(\mu_T^{t,\nu,u})$$

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**Theorem – Characterization [AH24]** Assume $f$ to be locally Lipschitz with linear growth, have convex $f[\mu, U]$, **weakly continuous** and $\mathfrak{J} : \mathscr{P}_2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ to be **weakly lsc**. Then

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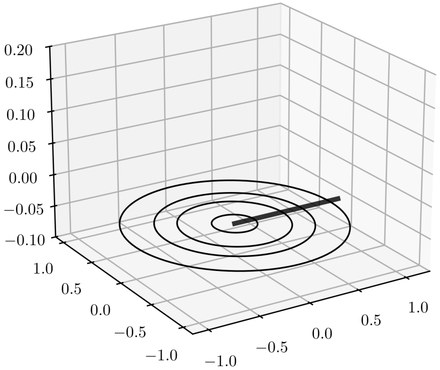
# Another idea: Lions' lift

⚠ Generalization of ideas from $\mathscr{P}_2$, mistakes are mine and not imputable to Lions or collaborators. ⚠

Assume $\Omega$ is isometric to the quotient of a Hilbert space $E$ by the action of a group of isometries. For any $\varphi : \Omega \to \mathbb{R}$, define its lift

$$\Phi(v) := \varphi([v]).$$



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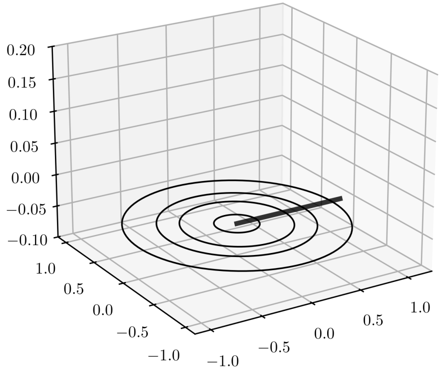
Define $\varphi$ to be differentiable at $x$ if $\Phi$ is Fréchet-differentiable in $E$ at some point of the equivalence class $x$.

- test functions $\sim$ subset of $\mathcal{C}^1(E)$



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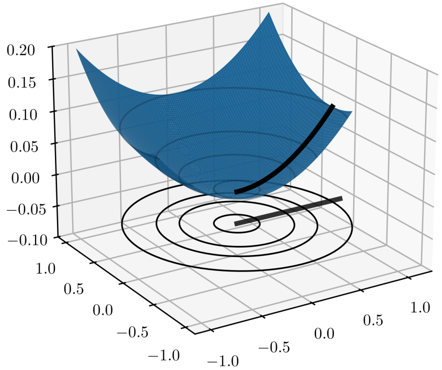

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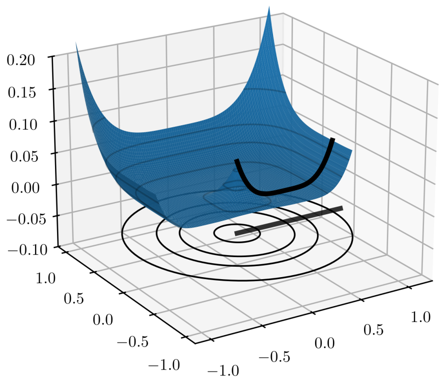

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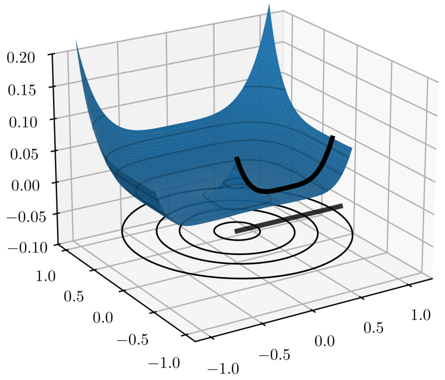

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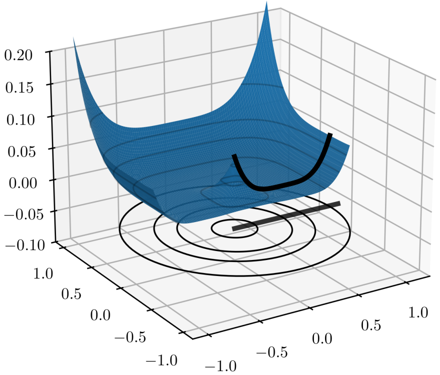

# Thank you!

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