A relaxation theorem in CAT(0) spaces

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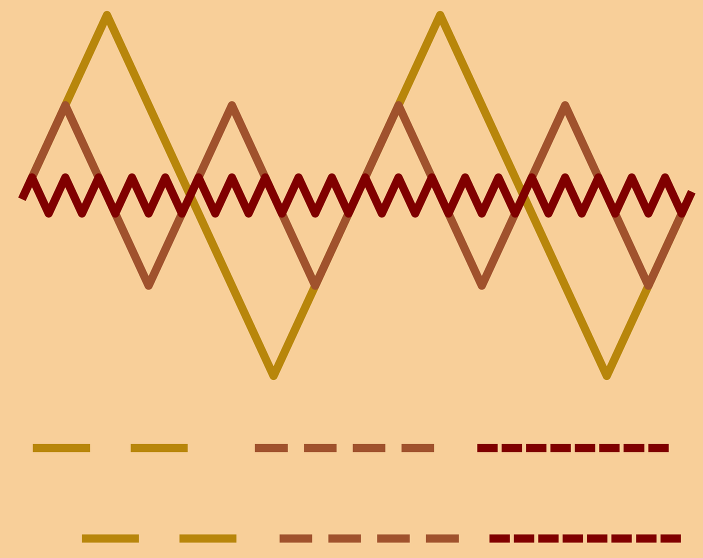
Motivation

Control problems are optimization problems set on the trajectories of a dynamical system. Among them, a Mayer problem writes as

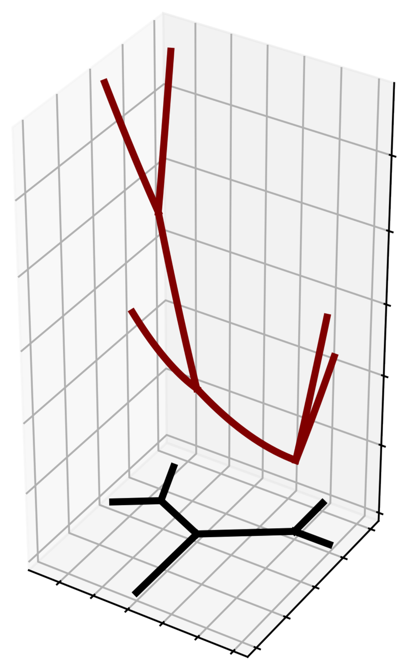
$$\text{Minimize } \mathfrak{J}\left(y_T^{0,x,u}\right) \text{ over all controls } u(\cdot) \in L^1(0,T;U), \tag{1}$$

where $\mathfrak{J}$ is a given terminal cost, U a compact set of controls, and the trajectory $\left(y_s^{t,x,u}\right)_{s\in[t,T]}$ is the solution of the controlled dynamical system $\dot{y}_s = f(y_s, u(s))$ issued from x at time t.
Our aim is to study such problems in the so-called *CAT(0) spaces*, i.e. metric geodesic spaces with a 2-convex squared distance. This poster focuses on the *existence* of an optimal control, which is usually obtained from compactness of the set of trajectories issued from a given point. In $\mathbb{R}^d$, this set is closed if $f(\cdot,U)$ is convex for each $x\in\Omega$. What would be a proper generalisation of this assumption in a CAT(0) space Ω?



Formulation

The first class of ODEs to be defined in metric spaces is the family of gradient flows. In a CAT(0) space Ω, Alexander, Kapovitch and Petrunin [AKP22] were able to show that any Lipschitz and semiconcave function $F:\Omega\to\mathbb{R}$ generates a semigroup $(h,x)\mapsto \mathrm{GF}_F(h,x)$ whose right derivative at all time coincides with the metric gradient of F.



To define a general ODE, consider a "potential dynamic" $F:\Omega\times U\to \mathrm{Lip}_{\mathrm{conv}}(\Omega;\mathbb{R})$ valued in convex and Lipschitz functions. The ODE formally writes as

$$\dot{y}_s = \nabla_{y_s}(-F(y_s,u(s))). \tag{2}$$

Definition. *A curve $(y_s)_{s\in[0,T]}$ is a solution of (2) if it is absolutely continuous, and for almost any $s\in[0,T]$, there holds*

$$\lim_{h\searrow 0} \frac{d\left(y_{s+h}, \mathrm{GF}_{-F(y_s,u(s))}(h,y_s)\right)}{h} = 0.$$

This formulation is inspired from the theory of *mutational equations*, and uses the results of existence, uniqueness and estimates for the solutions of Frankowska and Lorenz [FL22]. To control the variation of the dynamic, consider the seminorm on $\mathrm{Lip}_{\mathrm{conv}}(\Omega;\mathbb{R})$ given by

$$|\varphi|_\infty := \sup_{(x,v)\in \mathrm{T}\Omega,\ |v|_x=1} \left|D_x\varphi(v)\right|,$$

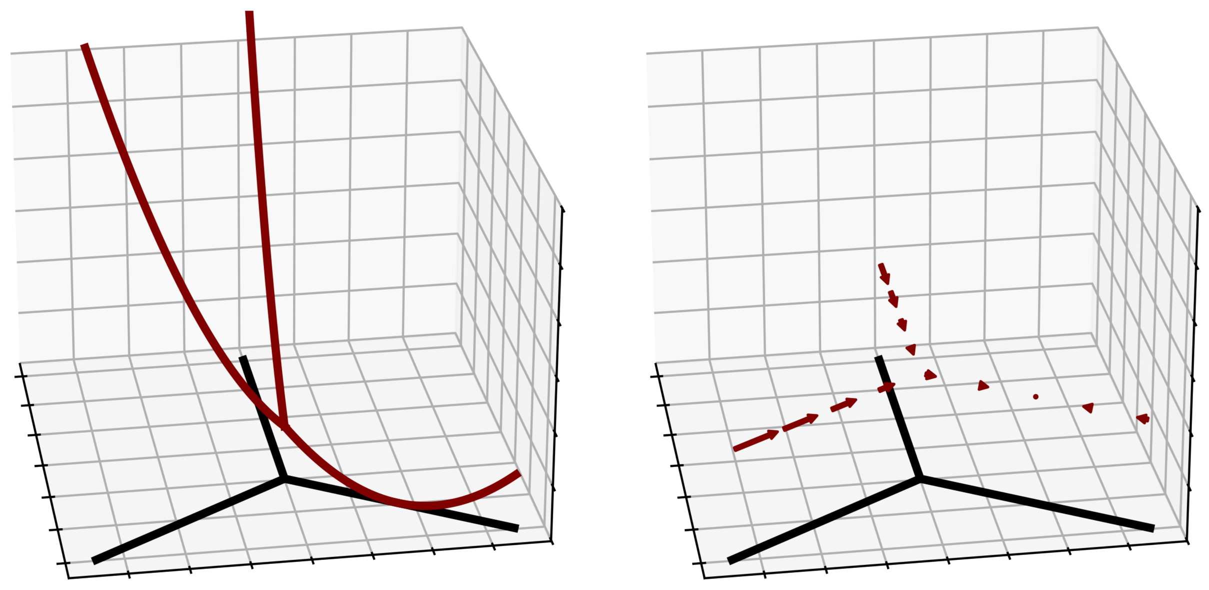
where $D_x\varphi(v)$ denotes the directional derivative of φ.

Proposition. *Assume that F is Lipschitz with respect to $|\cdot|_\infty$. Any control $u(\cdot)\in L^1(0,T;U)$ generates a unique flow of (2), which is continuous with respect to the initial point.*

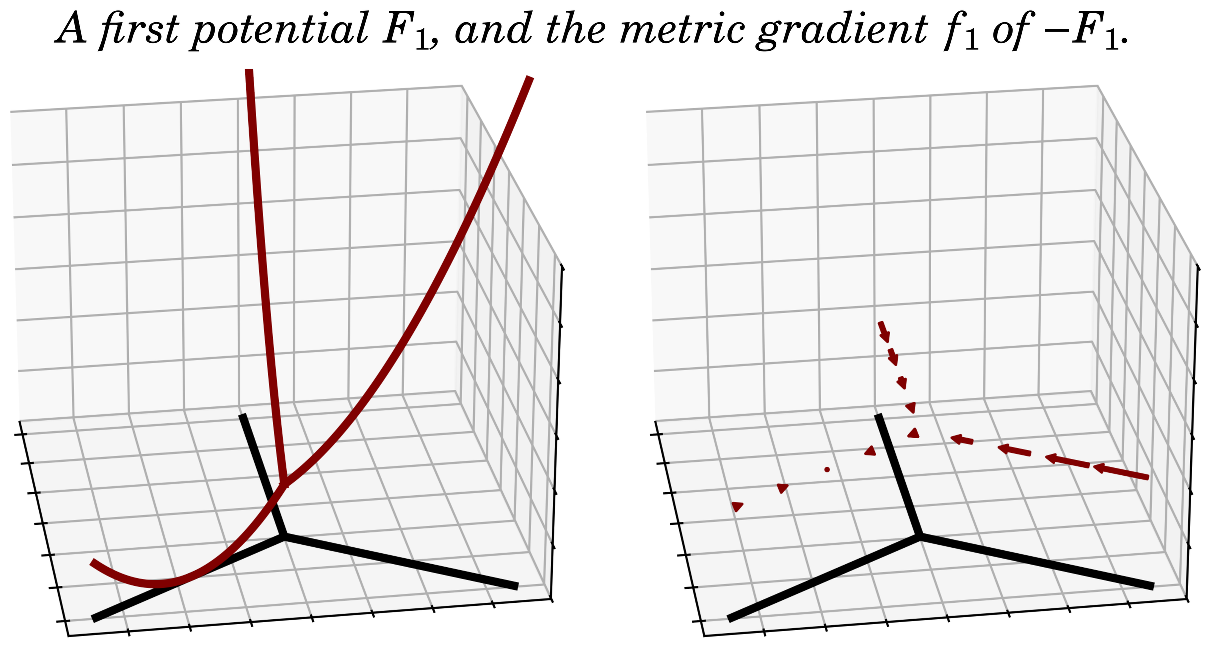
Theorem (Variational characterization)**.** Assume F to be Lipschitz with respect to $|\cdot|_\infty$. An absolutely continuous curve $(y_s)_{s\in[0,T]}$ is a solution of (2) if and only if for almost any $s\in[0,T]$, it satisfies the *Evolutionary Variational Inequality*

$$\frac{d}{ds}\frac{d^2(y_s,z)}{2} \leq F(y_s,u(s))(z) - F(y_s,u(s))(y_s) \qquad \forall z\in\Omega.$$

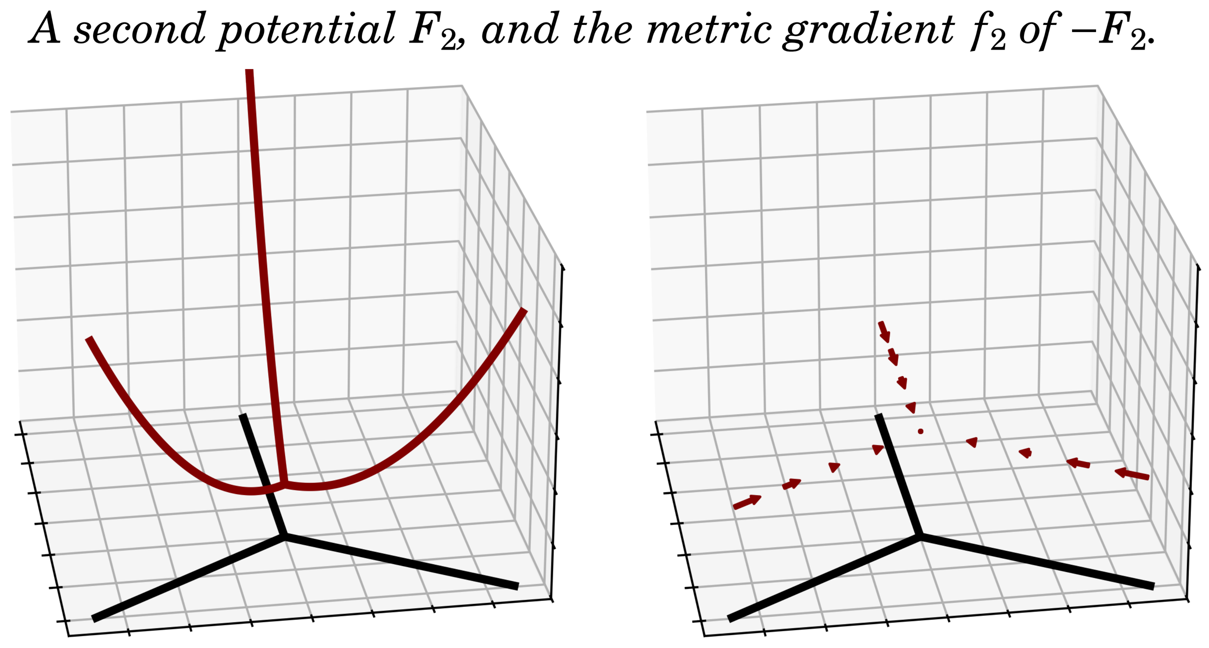
Relaxation



A first potential F_1, and the metric gradient f_1 of $-F_1$.



A second potential F_2, and the metric gradient f_2 of $-F_2$.



The potential $\frac{1}{2}F_1+\frac{1}{2}F_2$ and the metric gradient of its opposite. Observe that the latter is the geodesic mean of f_1 and f_2.

The formulation through a potential dynamic allows to embed $\left(\mathrm{Lip}_{\mathrm{conv}}(\Omega;\mathbb{R}), |\cdot|_\infty\right)$ in a Banach space, and take convex hulls therein. Denote $\mathscr{P}(U)$ the set of Borel probability measures on U.

Definition (Relaxed dynamic)**.** *Let $F:\Omega\times U\to Lip_{conv}(\Omega;\mathbb{R})$ be a potential dynamic with compact images. Its relaxed counterpart is given by*

$$\overline{co}F:\Omega\times\mathscr{P}(U)\to Lip_{conv}(\Omega;\mathbb{R}), \qquad \overline{co}F(x,\omega) := \int_{u\in U} F(x,u)\,d\omega(u).$$

One shows that the ODE (2) with second member $\overline{\mathrm{co}}F$ still admits a well-defined flow for any *relaxed control* $\omega(\cdot)\in L^1(0,T;\mathscr{P}(U))$, where $\mathscr{P}(U)$ is endowed with a Wasserstein distance.

Theorem (Closure of the set of trajectories)**.** Assume F to be Lipschitz with respect to $|\cdot|_\infty$. For any $x\in\Omega$, the closure of the set

$$\left\{ (y_s)_{s\in[0,T]} \;\middle|\; y_0 = x, \quad \dot{y}_s = \nabla_{y_s}(-F(y_s,u(s))) \text{ for some } u(\cdot)\in L^1(0,T;U) \right\}$$

in $\mathrm{AC}([0,T];\Omega)$ is given by

$$\left\{ (y_s)_{s\in[0,T]} \;\middle|\; y_0 = x, \quad \dot{y}_s = \nabla_{y_s}(-\overline{\mathrm{co}}F(y_s,\omega(s))) \text{ for some } \omega(\cdot)\in L^1(0,T;\mathscr{P}(U)) \right\}.$$

In particular, if F has convex images, it generates compact sets of trajectories. One could wonder if the convexification procedure on energies has any meaning at the level of vector fields. In $\mathbb{R}^d$, one may restrict the image of F in the set of linear potentials $y\mapsto\langle y,v\rangle$ for some $v\in\mathbb{R}^d$, so that

$$\overline{\mathrm{co}}F(x,\omega) = \int_{u\in U}\langle\cdot, v_{x,u}\rangle\, d\omega(u) = \langle\cdot, v_{x,\omega}\rangle, \qquad \text{where} \qquad v_{x,\omega} := \int_{u\in U} v_{x,u}\,d\omega(u).$$

Unfortunately, in a general CAT(0) space, there is no such linear functional, although a *metric scalar product* $\langle p,v\rangle_x := \frac{1}{2}\left(|p|_x^2 + |v|_x^2 - d_x^2(p,v)\right)$ is defined in each tangent cone $(\mathrm{T}_x\Omega, d_x)$. However:

Proposition. *If ω is concentrated on* Gâteaux-differentiable *functions, which satisfy*

$$D_x\varphi(v) = \langle\nabla_x\varphi, v\rangle_x \qquad \forall v\in \mathrm{T}_x\Omega,$$

then the gradient of the convex combination is the barycentre of the gradients in the sense of Sturm [Stu03], i.e.

$$\nabla_x\left(\int_{u\in U}\varphi_u\,d\omega(u)\right) = Bary_{\mathrm{T}_x\Omega}\nabla_x\varphi_{\pi_u}\#\omega.$$

This equality legitimates the use of a convex hull in the Banach space of potentials. The restriction to Gâteaux-differentiable maps is not stringent, since such functions may be built from the squared distance.

Conclusion

Assume that the potential dynamic F is Lipschitz with convex images. Then the Mayer control problem admits solutions. If F is not convex-valued, then the problem (1) can be *relaxed* by substituting the convexified dynamic $\overline{\mathrm{co}}F$ to F, without changing the optimal value. As in $\mathbb{R}^d$, optimal controls exist in the form of *Young measures* over U.

[AKP22] Stephanie Alexander, Vitali Kapovitch, and Anton Petrunin. Alexandrov geometry: Foundations, October 2022. Preprint (arXiv:1903.08539).

[FL22] Hélène Frankowska and Thomas Lorenz. Filippov's Theorem for Mutational Inclusions in a Metric Space, May 2022.

[Stu03] Karl-Theodor Sturm. Probability measures on metric spaces of nonpositive curvature. In *Contemporary Mathematics*, volume 338, pages 357–390. American Mathematical Society, 2003.

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