An Helmholtz-Hodge decomposition in the Wasserstein space

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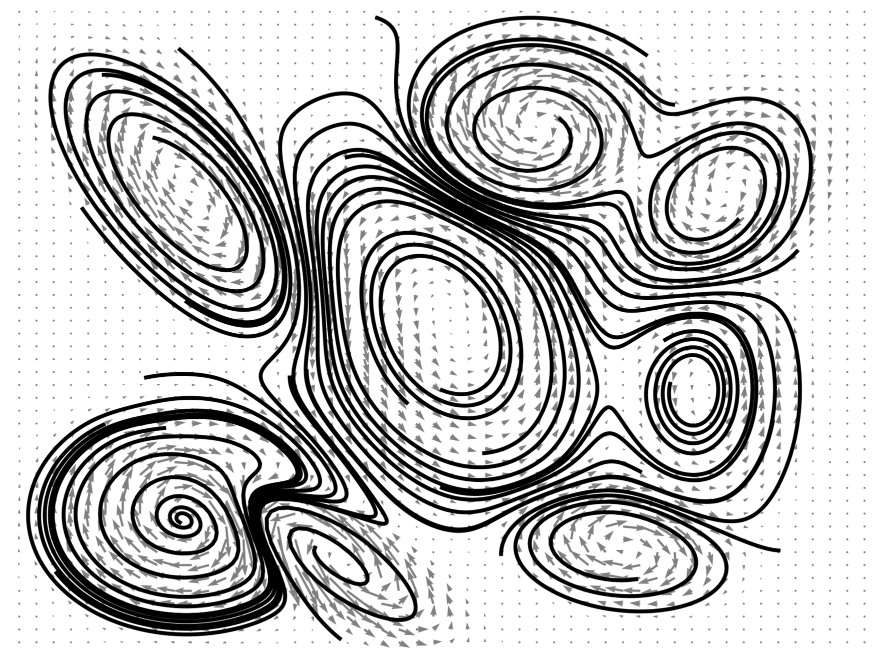
Picture yourself an incompressible fluid enclosed in a bounded domain. Assume we may "push" some part of the fluid. Being incompressible, this fluid particle will transmit its motion to its immediate neighbours, which will propagate their own motion as well, and so on inductively. As the domain is bounded, the fluid has no choice but to eventually *come back* to fill the volume left empty by the initial impulse. Our action essentially created interlaced swirls of varying shapes, so that the displacements mutually compensate. If we were able to follow the trajectory of a single droplet, we would see it coil and curl as to form a closed loop, and resemble the *solenoids* that one encounters in electromagnetics.
Let us switch now from a Lagrangian to an Eulerian point of view, and attach to each point of our domain the velocity of the nearby fluid. We obtain what is conveniently named a *solenoidal vector field*. These fields form a vector space, whose closure in the Hilbert space L^2 is the natural setting for the incompressible Navier-Stokes equation. It turns out that the orthogonal complement of this space is no less interesting, and recently attracted a lot of attention in the community of optimal transport.



A solenoidal field and some of its flow lines.

Let $\mathscr{P}_2(\mathbb{R}^d)$ be the set of Borel probability measures μ on $\mathbb{R}^d$ such that $\int_x |x|^2 d\mu$ is finite. If f were to belong to $L^2_\mu(\mathbb{R}^d;\mathrm{T}\mathbb{R}^d)$, the transport of μ by f would be defined as the measure $f\#\mu := \mu \circ f^{-1}$. By the pushforward operation $\#$, the mass of μ located at x is sent to the point $x+f(x)$. One could want to generalize this by allowing to split the mass in various directions v, each with a probability $\xi_x(v)$. This gives rise to "measure fields" $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, which are probability measures on the tangent space $\mathrm{T}\mathbb{R}^d := \{(x,v) \mid x \in \mathbb{R}^d,\ v \in \mathrm{T}_x\mathbb{R}^d\}$ representable by $\int_x \xi_x(\cdot) d\mu$ for some measurable family $(\xi_x)_x$. Thereon μ gets transported by ξ to the measure $(\pi_x + \pi_v)\#\xi$, which associates to each Borel set $A \subset \mathbb{R}^d$ the quantity $\int_{(x,v)\in\mathrm{T}\mathbb{R}^d} \mathbb{1}_A(x+v) d\xi$.

Wasserstein distance

The distance between $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ reads as

$$d^2_{\mathcal{W}}(\mu,\nu) := \inf \int_{(x,v)\in\mathrm{T}\mathbb{R}^d} |v|^2 d\xi(x,v),$$

where the infimum is taken over the measure fields $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ such that $(\pi_x+\pi_v)\#\xi = \nu$.

Cone distance

The distance between $\xi, \zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ is

$$W^2_\mu(\xi,\zeta) := \int_{x\in\mathbb{R}^d} d^2_{\mathcal{W},\mathrm{T}_x\mathbb{R}^d}(\xi_x,\zeta_x)\, d\mu(x).$$

The distance $W_\mu(\xi, 0_\mu)$ to the null velocity, that puts mass only on pairs $(x,0)$, is denoted $\|\xi\|_\mu$.

Metric scalar product

To each measure $\mu \in \mathscr{P}_2(\mathbb{R}^d)$ is attached a metric scalar product, given by

$$\langle \xi,\zeta\rangle_\mu := \frac{1}{2}\left[\|\xi\|^2_\mu + \|\zeta\|^2_\mu - W^2_\mu(\xi,\zeta)\right]$$

for any two measure fields $\xi,\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$.

The infimum defining $d_{\mathcal{W}}$ is reached over a set of measure fields that behave like gradients of semiconvex functions. On this account, we refer the reader to the enthousiastic presentation [San15]. As a matter of fact, the collection of these infima contains and extends the orthogonal of solenoidal fields in $L^2_\mu(\mathbb{R}^d;\mathrm{T}\mathbb{R}^d)$. In turn, solenoidal fields may be generalized through orthogonality with respect to $\langle\cdot,\cdot\rangle_\mu$. Let us present both as a diptych.

Tangent measure fields

For any $\alpha > 0$, denote by $\alpha\cdot\xi$ the rescaled measure field $\xi(\cdot/\alpha)$. Let

$$\mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d) := \overline{\left\{\alpha\cdot\xi \text{ s.t. } \alpha>0 \text{ and } \int_{(x,v)} |v|^2 d\xi = d^2_{\mathcal{W}}(\mu,(\pi_x+\pi_v)\#\xi)\right\}}^{W_\mu}.$$

This set, known as the *geometric tangent space* [Gig08], is characterized by the ability to escape from μ at maximal speed, in that

$$\xi \in \mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d) \quad\Longleftrightarrow\quad \lim_{h\searrow 0} \frac{d_{\mathcal{W}}(\mu,(\pi_x+\pi_v)\#(h\cdot\xi))}{h} = \|\xi\|_\mu.$$

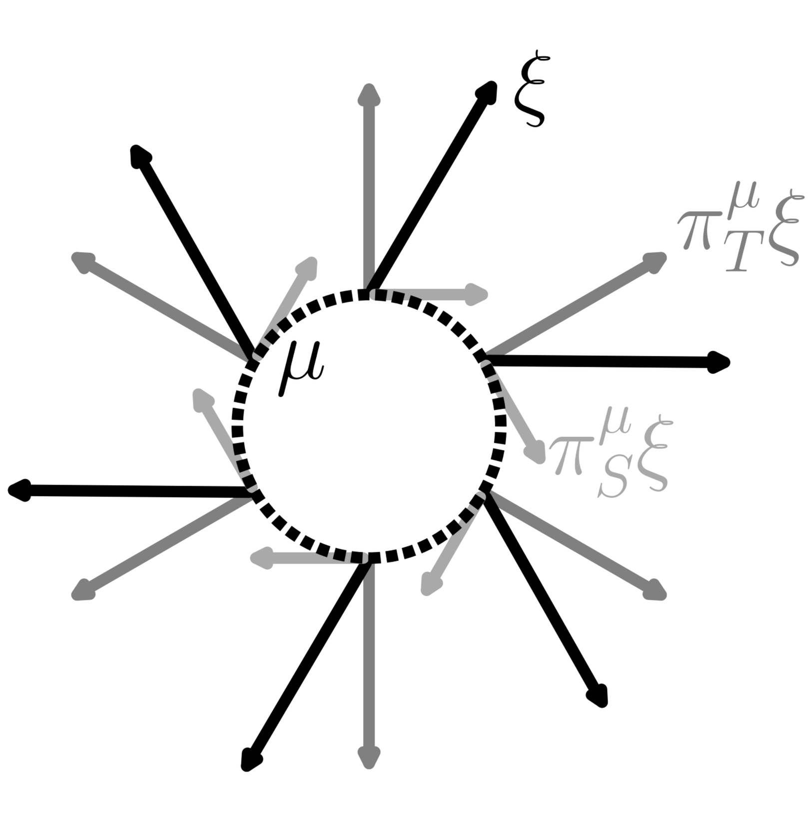
Solenoidal measure fields

A measure field $\zeta \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ is said to be *solenoidal* if

$$\langle\xi,\zeta\rangle_\mu = 0 \qquad \forall \xi \in \mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d).$$

We denote by $\mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d)$ the collection of such fields. It turns out that they are the fields spiraling around their initial point, in the sense that

$$\zeta \in \mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d) \quad\Longleftrightarrow\quad \lim_{h\searrow 0} \frac{d_{\mathcal{W}}(\mu,(\pi_x+\pi_v)\#(h\cdot\zeta))}{h} = 0.$$



A measure field and its projections.

Both sets $\mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ and $\mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d)$ enjoy projectors π^μ_T and π^μ_S, that to any measure field $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, associate the unique argument of the minimum of $W_\mu(\xi,\cdot)$ among their respective members. Figure 2 illustrates the projections when μ is concentrated on the unit circle $\mathbb{S}^1$. As a side remark, the set $\mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d)$ is upper semicontinuous in the set-valued sense with respect to μ in the Wasserstein topology of $\mathscr{P}_2(\mathrm{T}\mathbb{R}^d)$, while the geometric tangent space is lower semicontinuous. In other words, solenoidal measure fields are more stable with respect to the base measure than their tangent cousins.

We consider our main result the fact that any $\xi \in \mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$ decomposes in an unique way into its projection $\pi^\mu_T\xi$ over the tangent space and its projection $\pi^\mu_S\xi$ over the solenoidal space. If ξ were to be equal to $f\#\mu$ for some $f \in L^2_\mu(\mathbb{R}^d;\mathrm{T}\mathbb{R}^d)$, then the decomposition would take the form $f = g+h$, in which g is a solenoidal field and h belongs to the closure in L^2_μ of the gradients of smooth functions. When μ is the Lebesgue measure, this result is known as the Helmholtz-Hodge theorem [Lad87, Chap.1, Section 2]. In the general case, one shows that ξ is also given by a "sum" through the choice of a particular coupling between $\pi^\mu_T\xi$ and $\pi^\mu_S\xi$.

Some of the Hilbertian identities carry over in $\mathscr{P}_2(\mathrm{T}\mathbb{R}^d)_\mu$, and some does not. Under the notations of the previous paragraph, there always holds

$$\|\xi\|^2_\mu = \|\pi^\mu_T\xi\|^2_\mu + \|\pi^\mu_S\xi\|^2_\mu, \qquad \langle\xi,\eta\rangle_\mu = \langle\pi^\mu_T\xi,\eta\rangle_\mu \quad \forall\eta\in\mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d), \qquad \text{and} \qquad \langle\xi,\zeta\rangle_\mu = \langle\pi^\mu_S\xi,\zeta\rangle_\mu \quad \forall\zeta\in\mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d).$$

However, the Pythagoras formula $W^2_\mu(\xi,\zeta) = W^2_\mu(\pi^\mu_T\xi,\pi^\mu_T\zeta) + W^2_\mu(\pi^\mu_S\xi,\pi^\mu_S\zeta)$ does not hold in general. Even further, distinct measure fields may share the same projections over $\mathbf{Tan}_\mu\mathscr{P}_2(\mathbb{R}^d)$ and $\mathbf{Sol}_\mu\mathscr{P}_2(\mathbb{R}^d)$. Such details spice up the rich – and still mysterious – geometry of the Wasserstein space.

[Gig08] Nicola Gigli. *On the Geometry of the Space of Probability Measures Endowed with the Quadratic Optimal Transport Distance*. PhD thesis, Scuola Normale Superiore di Pisa, Pisa, 2008.
[Lad87] Ol'ga A. Ladyženskaja. *The Mathematical Theory of Viscous Incompressible Flow*. Number 2 in Mathematics and Its Applications. Gordon and Breach, 1987.
[San15] Filippo Santambrogio. *Optimal Transport for Applied Mathematicians*, volume 87 of *Progress in Nonlinear Differential Equations and Their Applications*. Springer International Publishing, Cham, 2015.

JDD2024, Joint PhD Student's day between MIIS and PSIME doctoral schools, 11 June 2024, Rouen, France.